

Some controllability results for the relativistic Vlasov-Maxwell system

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RÉSUMÉ. L'objectif de cette note est de présenter les résultats récents concernant la contrôlabilité du système de Vlasov-Maxwell, qui sont prouvés dans le papier [10] écrit en collaboration avec Olivier Glass.

Quelques résultats de contrôlabilité pour le système de Vlasov-Maxwell relativiste

ABSTRACT. The goal of this note is to present the recent results concerning the controllability of the Vlasov-Maxwell system, which are proved in the paper [10] by Olivier Glass and the author.

1. Introduction

We consider the relativistic Vlasov-Maxwell system set in the two-dimensional torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$, with a source supported in an open subset $\omega \subset \mathbb{T}^2$:

$$\left\{ \begin{array}{l} \partial_t f + \hat{v} \cdot \nabla_x f + \operatorname{div}_v \left[\left(E + \frac{1}{c} \hat{v}^\perp B \right) f \right] = \mathbb{1}_\omega G, \quad t > 0, x \in \mathbb{T}^2, v \in \mathbb{R}^2, \\ \partial_t E_1 - c \partial_{x_2} B = - \int_{\mathbb{R}^2} f \hat{v}_1 dv, \quad \partial_t E_2 + c \partial_{x_1} B = - \int_{\mathbb{R}^2} f \hat{v}_2 dv, \\ \partial_t B + c \operatorname{rot} E = 0, \\ \operatorname{div} E = \int_{\mathbb{R}^2} f dv - \int_{\mathbb{R}^2 \times \mathbb{T}^2} f dv dx, \\ f|_{t=0} = f_0, \quad E|_{t=0} = E_0, \quad B|_{t=0} = B_0, \end{array} \right. \quad (1.1)$$

where

$$\hat{v} := \frac{v}{\sqrt{1 + \frac{|v|^2}{c^2}}}$$

is the relativistic velocity and $c > 0$ is the *speed of light*. The distribution function $f(t, x, v)$ describes the statistical distribution of a population of electrons in a collisionless plasma: the quantity $f(t, x, v) dx dv$ can be interpreted as the density at time t whose position is close to x and velocity close to v . As usual, $E(t, x)$ stands for the electric field (here, this is a two-dimensional vector) and $B(t, x)$ for the magnetic field (here this is a scalar quantity). They are solutions to the Maxwell equations, with some sources ($\rho := \int f dv$, $j := \int f \hat{v} dv$), which means that the electromagnetic fields are induced by the electrons themselves. The electromagnetic fields act on the motion of the electrons through the Lorentz force $E + \frac{1}{c} \hat{v}^\perp B$ (where \hat{v}^\perp stands for \hat{v} rotated of $\pi/2$).

Finally, $\mathbb{1}_\omega G$ is a source in the Vlasov equation, which can be interpreted as an absorption or creation of charged particles.

We are interested in the *controllability* properties of the Vlasov-Maxwell system, by means of an interior control localized in ω . The basic control question is the following:

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given (f_1, E_1, B_1) a target and some control time $T > 0$, is it possible to choose G such that the (non-linear) dynamics can be driven to:

$$f|_{t=T} = f_1, \quad E|_{t=T} = E_1, \quad B|_{t=T} = B_1 ? \quad (1.2)$$

The investigation of such a question for other non-linear kinetic equations has already been done in the context of the Vlasov-Poisson equation by O. Glass [8] and for the Vlasov-Poisson equation with external force fields by O. Glass and the author [11]. The latter case can be somehow considered as a toy model for the Vlasov-Maxwell case, and will play an important role for one of our next results.

As it is well-known, the Maxwell equations are basically a system of two coupled (linear) wave equations. Concerning the controllability of wave equations, we have a famous result of Rauch-Taylor [14], and Bardos-Lebeau-Rauch [3], which involves some geometric condition on the control set ω , that we can introduce now:

Definition 1.1. *The open subset ω of \mathbb{T}^2 satisfies the Geometric Control Condition (GCC) if:*

$$\begin{aligned} & \text{There exists } T > 0 \text{ such that for any } x \in \mathbb{T}^2 \text{ and any direction } e \in \mathbb{S}^1, \\ & \text{there exists } y \in [0, T] \text{ such that } x + ye \in \omega. \end{aligned} \quad (1.3)$$

Loosely speaking, this means that any ray of light meets the control set in finite time, and that all the information can be “observed” from ω . We have:

Theorem 1.2 (Rauch and Taylor). *Assume that $\omega \subset \mathbb{T}^2$ satisfies GCC. Let $(\varphi_0, \Psi_0), (\varphi_1, \Psi_1) \in H^1(\mathbb{T}^2) \times L^2(\mathbb{T}^2)$ some initial and final data. Then there exists $G \in L^2(\mathbb{T}^2)$ such that the solution φ to the wave equation:*

$$\begin{cases} \partial_t^2 \varphi - \Delta_x \varphi = \mathbb{1}_\omega G, \\ \varphi|_{t=0} = \varphi_0, \quad \partial_t \varphi|_{t=0} = \Psi_0, \end{cases}$$

satisfies:

$$\varphi|_{t=T} = \varphi_1, \quad \partial_t \varphi|_{t=T} = \Psi_1.$$

As shown in [3] and [4], GCC is not only a sufficient condition to get controllability for the wave equation, it is also necessary. In view of these facts, it is natural for our problem (whose structure is transport + waves) to assume GCC on the control set ω . Indeed, we have the following result:

Theorem 1.3 (Glass and Han-Kwan). *We assume that ω satisfies the geometric control condition. There exists $T_0 > 0$, such that for any $T > T_0$ if for $i = 0, 1$, $(f_i, E_i, B_i) \in H^3(\mathbb{T}^2 \times \mathbb{R}^2) \times H^3(\mathbb{T}^2) \times H^3(\mathbb{T}^2)$ are such that f_i is compactly supported in v and satisfy the compatibility conditions:*

$$\operatorname{div} E_i = \int_{\mathbb{R}^2} f_i dv - \int_{\mathbb{T}^2 \times \mathbb{R}^2} f_i dv dx, \quad (1.4)$$

$$\int_{\mathbb{T}^2 \times \mathbb{R}^2} f_0 dv dx = \int_{\mathbb{T}^2 \times \mathbb{R}^2} f_1 dv dx, \quad (1.5)$$

as well as the smallness assumption

$$\|(f_i, E_i, B_i)\|_{H^3} \leq \kappa, \quad i = 0, 1, \quad (1.6)$$

for κ small enough, then there exists a control $G \in H^2([0, T] \times \mathbb{T}^2 \times \mathbb{R}^2)$, supported in ω , which drives the Vlasov-Maxwell system from (f_0, E_0, B_0) to (f_1, E_1, B_1) .

This theorem is a small data result: one speaks of a local controllability property. The limitation on the control time is expected, because of the finite speed of propagation of information in the Maxwell equations. It would be interesting though to find the minimal time of control in this result.

The following of this note is organized as follows: Section 2 is first devoted to a sketch of the proof of this latest result. In Section 3, we will present and explain another result, maybe more surprising, for a control set ω which does not satisfy the geometric control condition.

2. Sketch of the proof of Theorem 1.3

2.1. Preliminary 1: The return method

In order to prove the Theorem, which concerns small data, the first natural idea is to follow the proof scheme:

- Linearize the equations around the trivial state $(0, 0, 0)$ and prove some controllability properties of these linear equations.
- Show that these properties are somehow preserved after some small perturbation and finally prove by a fixed point argument that this allows to give a solution to the non-linear problem.

Unfortunately, we are in a situation where this usual proof scheme fails. Indeed, after linearization around the trivial state, we obtain the following relativistic free transport equation:

$$\begin{cases} \partial_t f + \hat{v} \cdot \nabla_x f = \mathbb{1}_\omega(x)G(t, x, v), \\ f_{t=0} = f_0, \end{cases}$$

which fails to be controllable in general. By Duhamel's formula, we have the representation (characteristics are straight lines):

$$f(t, x, v) = f_0(x - t\hat{v}, v) + \int_0^t \mathbb{1}_\omega(x - (t - \tau)\hat{v})G(\tau, x - (t - \tau)v, v) d\tau.$$

To be able to influence the value of f at some time $T > 0$, it clearly necessary that $\mathbb{1}_\omega(x - (T - \tau)\hat{v})$ is different from 0 for any $(x, v) \in \mathbb{T}^2 \times \mathbb{R}^2$, at least for some $\tau \in [0, T]$. From this remark, we deduce that there is an obstruction to controllability coming from the geometry of the trajectories. It could be that for any $t \in \mathbb{R}$, $x + t\hat{v}$ never meets ω : this is the *bad direction* obstruction. Of course, this does not occur when ω satisfies GCC, but we have to keep in mind this problem, in view of the second result that will be discussed in the last section. Another obstruction comes from the fact that the velocity v can have an arbitrarily small modulus, so that even if $x + t\hat{v}$ always meets ω for some $t \in \mathbb{R}$, this can be only for extremely large values of $|t|$: this is the *slow particles* obstruction.

In order to circumvent this difficulty, we shall rely on the return method of J.-M. Coron (we refer to the book [5] for many other applications, especially for the incompressible Euler equations, and to [9] for another pedagogical presentation). The principle of this method is the following: instead of linearizing around the trivial state, we look for a particular homoclinic solution to the full Vlasov-Maxwell system (with some source supported in ω), starting from $(0, 0, 0)$ and reaching again $(0, 0, 0)$ after some time $T > 0$, but which can be highly non-trivial inside the interval of time. We ask that the linearized equations around this solution enjoy nice controllability properties.

In the kinetic context, we look for a solution $(\bar{f}, \bar{E}, \bar{B})$ to Vlasov-Maxwell system, with a suitable source supported in ω , starting from $(0, 0, 0)$ and returning to $(0, 0, 0)$ such that the characteristics

$$\begin{cases} \frac{dX}{dt}(t, x, v) = \hat{V}(t, X(t, x, v)) \quad \text{and} \quad X(0, x, v) = x, \\ \frac{dV}{dt}(t, x, v) = \bar{E}(t, X) + \frac{\hat{V}^\perp}{c} \bar{B}(t, X) \quad \text{and} \quad V(0, x, v) = v, \end{cases}$$

satisfy, for some large enough $T > 0$:

Any trajectory X meets the control zone ω during $[0, T]$.

This will be sufficient for proving some results for the full non-linear equations. In view of the future fixed point scheme, we will actually need a slightly stronger property: let $\omega' \subset \omega'' \subset \omega$ where ω', ω'' are open sets still satisfying GCC, then we ask that

Any trajectory X meets the zone ω' during $[0, T]$.

2.2. Preliminary 2: Control of the Maxwell equation with GCC

In order to implement this strategy, we will rely on some controllability results for the Maxwell equations. As already mentioned, Maxwell equations can be recast as a system of coupled linear wave equations:

$$\begin{cases} \partial_t^2 E - c^2 \Delta_x E = -c^2 \nabla \rho - \partial_t j, \\ \partial_t^2 B - c^2 \Delta_x B = c \operatorname{rot} j. \end{cases}$$

Control is expected to be obtained by the source (ρ, j) , which is supported in ω . One difficulty is that this control has to satisfy, for consistency reasons, the local conservation of charge:

$$\partial_t \rho + \nabla \cdot j = 0. \quad (2.1)$$

This was overcome by Phung in [13], who showed that it is possible to control the system by using only divergence free currents j :

Theorem 2.1. *Assume that ω satisfies GCC. Let us assume that the control time $T > 0$ is large enough. Let $k \in \mathbb{N}^*$. For any $(E_0, B_0, E_1, B_1) \in H^k(\mathbb{T}^2)^4$, with $\operatorname{div} E_0 = \operatorname{div} E_1 = 0$, there exists a control function $\tilde{j} \in \cap_{s=0}^k H^s([0, T], H^{k-s}(\mathbb{T}^2))$ satisfying $j_{t=0} = j_{t=T} = 0$, such that for all $t \in [0, T]$, $\operatorname{div} \tilde{j} = 0$ and such that the solution (E, B) to the system:*

$$\begin{cases} \partial_t \tilde{B} + c \operatorname{rot} \tilde{E} = 0, & \partial_t \tilde{E} + c \operatorname{rot} \tilde{B} = -\tilde{j} \mathbb{1}_\omega, \\ \operatorname{div} \tilde{E} = 0, & \operatorname{div} \tilde{B} = 0, \\ \tilde{E}|_{t=0} = E_0, & \tilde{B}|_{t=0} = B_0, \end{cases} \quad (2.2)$$

satisfies $\tilde{E}|_{t=T} = E_1$, $\tilde{B}|_{t=T} = B_1$.

Actually, Phung deals with the much more complicated case of domains of \mathbb{R}^3 with boundaries, and with low regularity data. In our problem, it is important to deal with smooth electromagnetic fields and with a smooth control function: this follows for instance from arguments developed in [7].

2.3. Step 1: Construction of the reference solution

We can now describe the formal construction of a relevant reference solution:

1. As a first step, we take $(0, 0, 0)$ as reference solution. Observing that straight lines with “large” velocities have to “quickly” meet ω' , we find some time $T_0 > 0$ such that:

$$\forall x \in \mathbb{T}^2, \forall v \in \mathbb{R}^2, |v| \gg 1, \quad \bar{X}(t, x, v) \text{ meets } \omega' \text{ during } [0, T_0].$$

There remains to take care of particles whose initial velocity is not large enough.

2. To that purpose, we use Theorem 2.1: we accordingly find a smooth j_1 such that $\nabla \cdot j_1 = 0$, supported in ω , which drives the Maxwell equations from $(0, 0)$ to $(E_1 :=$

$(1, 1), 0)$ in some time T_1 . From that current, it is easy to find a solution to the Vlasov-Maxwell equation, with some source supported in ω . Indeed, let $Z_i(v) \geq 0$ (for $i = 1, 2$) be a smooth function such that $\int Z_i(v) dv = 0$ and $\int Z_i(v) \hat{v} dv = (\delta_{i=1}, \delta_{i=2})$. We set:

$$\bar{f}(t, x, v) := Z_1(v) j_1^1(t, x) + Z_2(v) j_1^2(t, x), \quad \text{with } j_1 = (j_1^1, j_1^2)$$

and (\bar{E}, \bar{B}) the solution to Maxwell with sources given by $\bar{\rho} = 0$ and $\bar{j} = j_1$. Then we observe that $(\bar{f}, \bar{E}, \bar{B})$ is a solution to Vlasov-Maxwell with a suitable source in ω . Note indeed that

$$\bar{G} := \partial_t \bar{f} + \hat{v} \cdot \nabla_x \bar{f} + \left(\bar{E} + \frac{\hat{v}^\perp}{c} \bar{B} \right) \cdot \nabla_v \bar{f}$$

is supported in ω , by definition of \bar{f} and j_1 . Note then that the state reached at time $T_0 + T_1$ is exactly $(0, E_1, 0)$, which happen to be a stationary solution to the Vlasov-Maxwell system (without any source). The effect of the electric field E_1 is to accelerate all particles; because the solution is stationary, we can wait enough time so that the velocity of particles exceeds that needed in the previous step. This implies that for some large $T_2 > 0$, we have the property:

$$\forall x \in \mathbb{T}^2, \forall v \in \mathbb{R}^2, \quad \bar{X}(t, x, v) \text{ meets } \omega' \text{ during } [0, T_2].$$

3. Finally, with a similar procedure as before, we bring back the system to the trivial state $(0, 0, 0)$.

2.4. Step 2: Absorption and fixed point procedure

The last step consists of a relevant fixed point scheme. Let us first describe how we define our fixed point operator \mathcal{V} . Let $g \in H^3$ be some distribution function close to the reference solution $(\bar{f}, \bar{E}, \bar{B})$ and satisfying the local conservation of charge.

We first define (E^g, B^g) the solution to the Maxwell equations with initial conditions (E_0, B_0) and some sources $(\rho^g, j^g) := (\int g dv, \int g \hat{v} dv)$. Then we define f_1 in the following way:

$$\begin{cases} f_1(t=0) = f_0, \\ \partial_t f_1 + \hat{v} \cdot \nabla_x f_1 + \left(E^g + \frac{\hat{v}^\perp}{c} B^g \right) \cdot \nabla_v f_1 = 0, \\ \text{Absorption procedure on } \partial\omega''. \end{cases} \quad (2.3)$$

For the sake of simplicity, we have chosen not to write precisely the absorption procedure (but it can be found in full details in [10]). Loosely speaking, the principle is to absorb certain particles who enter ω'' . This creates some new difficulties compared to the way the ‘‘usual’’ Cauchy problem is solved (see [16, 17, 1, 12] and many others).

Then, we only consider the restriction of f_1 on $\mathbb{T}^2 \setminus \bar{\omega}''$ and use an extension theorem to fill in smoothly the data inside ω'' . In this procedure, we have to be careful of preserving the local conservation of charge inside ω'' . One finally obtains a distribution function f_2 defined on $\mathbb{T}^2 \times \mathbb{R}^2$. We finally define:

$$\mathcal{V}(g) := \bar{f} + f_2. \quad (2.4)$$

The goal is then to use Schauder’s theorem to obtain a fixed point to this operator. To this end, we need in particular to get H^3 estimates for $\mathcal{V}(g)$. The main difficulty comes from the discontinuity of gradients across $\partial\omega''$, which is created because of the absorption procedure. We therefore have to perform H^3 estimates for the restriction of f_1 in ω'' and in $\mathbb{T}^2 \setminus \bar{\omega}''$ separately. When doing so, it turns out that many boundary terms coming from integration by parts have bad signs in view of energy estimates. To cure this problem, we have to make the sum of the estimates for each region, which allows to kill all terms with

a bad sign. In this procedure, we use several times the Vlasov equation satisfied in each region to trade normal derivatives (to $[0, T] \times \omega'' \times \mathbb{R}^2$) for tangential derivatives. We refer to the paper for more details.

Let us now denote (f, E, B) a fixed point to \mathcal{V} : this is a solution to Vlasov-Maxwell with a suitable source supported in ω , which is by construction close to the reference one. By a Gronwall argument, this implies that the characteristics (X, V) associated to (E, B) satisfy also:

Any trajectory X meets ω' during $[0, T]$.

Because of the absorption procedure, this means that $f \equiv 0$ outside ω'' . After some minor further manipulations (inside ω , we can do anything to modify the solution), this yields a solution which goes from (f_0, E_0, B_0) to $(0, 0, 0)$ with a source supported in ω .

Likewise we can build a solution which goes from (f_1, E_1, B_1) to $(0, 0, 0)$. Relying on the reversibility of the equations (if f is a solution, then $f(-t, x, -v)$ also is), this gives a solution which goes from $(0, 0, 0)$ to (f_1, E_1, B_1) . We finally obtain a solution which goes from (f_0, E_0, B_0) to (f_1, E_1, B_1) , with a source supported in ω (which is the control function). This puts an end to the proof of Theorem 1.3.

3. Another control result without Geometric Control Condition

Without GCC, it seems hopeless to be able to prove any controllability property for the full system, because of the Maxwell part in the equations. The idea behind the following result is that actually, we have more flexibility for what concerns the Vlasov equation. Thus, if we focus on the distribution function only (and abandon the idea of controlling the EM fields), the following result can be proved:

Theorem 3.1 (Glass and Han-Kwan). *Assume that ω contains a strip. Let $(f_0, E_0) \in H^3(\mathbb{T}^2 \times \mathbb{R}^2) \times H^3(\mathbb{T}^2)$ be some initial data and $f_1 \in H^3(\mathbb{T}^2 \times \mathbb{R}^2)$ a target such that f_0, f_1 are compactly supported in v and satisfying the compatibility conditions:*

$$\operatorname{rot} E_0 = 0, \quad \operatorname{div} E_0 = \int_{\mathbb{R}^2} f_0 dv - \int_{\mathbb{T}^2 \times \mathbb{R}^2} f_0 dv dx, \quad (3.1)$$

$$\int_{\mathbb{T}^2 \times \mathbb{R}^2} f_0 dv dx = \int_{\mathbb{T}^2 \times \mathbb{R}^2} f_1 dv dx. \quad (3.2)$$

Let $b_0 \in H^3(\mathbb{T}^2)$ be a magnetic field of the form b_0 such that $\int_{\mathbb{T}^2} b_0 \neq 0$. There exist $T_0 > 0$ such that for any $T > T_0$, there exists $c_T > 0$ such that for any $c > c_T$, if the following holds

$$\|(f_0, E_0)\|_{H^3} \leq \kappa, \quad \|f_1\|_{H^3} \leq \kappa, \quad (3.3)$$

$$B_0 = c b_0, \quad (3.4)$$

with κ small enough, then there exists a control G such that the solution (f, E, B) to the Vlasov-Maxwell system with initial data (f_0, E_0, B_0) satisfies:

$$f_{t=T} = f_1. \quad (3.5)$$

The strategy to prove this result is different to that of the previous theorem. The main difference lies in the construction of a reference solution. Indeed, in the absence of GCC, no exact controllability result is available for the Maxwell equations. Instead, the strategy we follow is inspired by the behaviour of the Vlasov-Maxwell system as the speed of light c goes to infinity. Some famous results (see [2, 6, 15]) indeed state that in the limit $c \rightarrow +\infty$, solutions to the Vlasov-Maxwell system are solutions to the Vlasov-Poisson system. The condition (3.4), which may seem strange otherwise, corresponds to the right scaling if one wants to recover a non-trivial magnetic field in the limit.

3.1. An approximation lemma

The key result is the following lemma, which states that the solutions to the Maxwell equations are well approximated, as c gets large by an electric field which is solution to a Poisson equation and the initial magnetic field (actually a more general version is proved in [10]).

Lemma 3.2. *Let $d = 2, 3$. Let E_0, j, ρ some C^∞ functions and B_0 a constant and uniform magnetic field. Let us consider the solution (E, B) to the Maxwell equations:*

$$\begin{cases} \partial_t B + c \operatorname{rot} E = 0, & \partial_t E + c \operatorname{rot} B = -j, \\ \operatorname{div} E = \rho - \int \rho dx, & \operatorname{div} B = 0, \\ E|_{t=0} = E_0, & B|_{t=0} = B_0. \end{cases} \quad (3.6)$$

At initial time we assume that the compatibility conditions are satisfied:

$$\operatorname{rot} E_0 = 0, \quad \operatorname{div} E_0 = \rho(0) - \int \rho(0) dx. \quad (3.7)$$

For all times we assume that the local conservation of charge is satisfied:

$$\forall x \in \mathbb{T}^d, \quad \partial_t \rho + \nabla_x \cdot j = 0, \quad (3.8)$$

as well as the zero-mean current property:

$$\int_{\mathbb{T}^d} j dx = 0. \quad (3.9)$$

Let E_∞ be the solution to the Poisson equation:

$$\begin{cases} \operatorname{rot} E_\infty = 0 \\ \operatorname{div} E_\infty = \rho - \int \rho dx. \end{cases} \quad (3.10)$$

Then, we have:

$$\begin{aligned} \|B - B_0\|_{L^\infty([0,t] \times \mathbb{T}^d)} &\leq \frac{C_{\rho,j} t}{c}, \\ \|E - E_\infty\|_{L^\infty([0,t] \times \mathbb{T}^d)} &\leq \frac{C'_{\rho,j} t}{c}, \end{aligned} \quad (3.11)$$

where $C_{\rho,j}$ and $C'_{\rho,j}$ are explicit constants depending only on ρ and j .

3.2. The reference solution in the Vlasov-Poisson case

This suggests that instead of the Maxwell case, we should first study the Poisson case, which is much more tractable (for instance, there is an infinite speed of propagation of information). We look for a reference solution (\bar{f}, \bar{E}) to the Vlasov-Poisson system with an external magnetic field B_0 , such that the characteristics (\bar{X}, \bar{V}) associated to $\bar{E} + \hat{v}^\perp B_0/c$ satisfy:

Any trajectory X meets the control zone $B(x_0, r_0)$ during $[0, T]$.

where $B(x_0, r_0)$ is a small ball contained in ω .

The construction is very similar to that of [11], except from the fact that some additional consistency relations have to be satisfied in order to apply our approximation lemma (this leads to some serious technical difficulties).

As before, there is a first obstruction coming from slow velocities. This can be solved as follows in the Poisson case. We denote by D a line of \mathbb{T}^2 , which does not cut the control zone ω (reduce ω if necessary) and n a unit vector, orthogonal to D . We have the proposition:

Proposition 3.3. *If ω contains a strip, there exists $\theta \in C^\infty(\mathbb{T}^2; \mathbb{R})$ such that*

$$\begin{aligned} \Delta\theta &= \rho \text{ in } \mathbb{T}^2, \\ \text{Supp } \rho &\subset \omega, \\ \forall x \in \mathbb{T}^2 \setminus \omega, \quad |\nabla\theta(x)| &> 0, \\ \int_D \nabla\theta \cdot n \, dx &= 0. \end{aligned}$$

The proof of this result, which can be extrapolated from [8], relies essentially on complex analysis tools. The third condition expresses that the electric field $\nabla\theta$ allows to accelerate particles. The fourth condition will be actually crucial to impose some consistency relations and is obtained thanks to the strip contained in ω .

Contrary to the case with GCC, we also have to cure the obstruction coming from *bad directions*. Fortunately, one can first observe that there is only a finite number of directions v_i of \mathbb{S}^1 such that :

$$\exists x \in \mathbb{T}^2, \forall t \in \mathbb{R}^+, x + tv_i \notin B(x_0, r_0).$$

To circumvent that defect, we will rely on an effect provided by the magnetic fields satisfying the bending condition (which was first exhibited in [11]). This “bending effect” can be easily understood in the baby-model where $c = +\infty$ and $b_0 \equiv 1$. In that simple case, the equations of characteristics read

$$\frac{dX}{dt} = V, \quad \frac{dV}{dt} = V^\perp, \quad X_{t=0} = x, \quad V_{t=0} = v.$$

and the solutions are explicit:

$$V = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} v, \quad X = \begin{pmatrix} \sin t & \cos t - 1 \\ 1 - \cos t & \sin t \end{pmatrix} v + x.$$

We are concerned here with large enough velocities. We distinguish between the two possibilities:

- **Good initial direction:** $v \neq v_i$

Loosely speaking, when $|v|$ is very large, the curvature of the circles is very small and the trajectories look very much like straight lines. One can show a good approximation by straight lines during a time of order $\mathcal{O}(\frac{1}{|v|})$, which means that X meets $B(x_0, r_0)$.

- **Bad initial direction:** $v = v_i$

The idea is to rely on the rotation induced by the magnetic field. After some time τ of order $\mathcal{O}(1)$, we observe that $V_{t=\tau}$ is not a bad direction anymore. At this point, we are as in the previous case, which means that X meets $B(x_0, r_0)$.

This effect holds for quite general magnetic fields, which satisfy a property which we call the bending condition.

Definition 3.4. *We say that a magnetic field b satisfies the bending condition if b or $-b$ satisfies the conditions 1) and 2).*

1) Geometric control condition. We assume that there exists a compact set K of \mathbb{T}^2 on which $b > 0$ and which satisfies the geometric control condition:

$$\begin{aligned} \text{For any } x \in \mathbb{T}^2 \text{ and any direction } e \in \mathbb{S}^1, \\ \text{there exists } y \in \mathbb{R}^+ \text{ such that } x + ye \in K. \end{aligned} \quad (3.12)$$

By compactness, we can define D the maximal time a geodesic can spend outside of K and d the minimal time a geodesic spends inside K . This allows us to introduce the second condition on the magnetic field:

2) Bound from below. We assume that there exists $\tilde{b} \in \mathbb{R}$ such that

$$\begin{aligned} b &\geq \tilde{b} \text{ on } \mathbb{T}^2, \\ D\tilde{b} + \frac{d}{2}\underline{b} &> 0. \end{aligned} \tag{3.13}$$

We refer to [11, 10] for detailed proofs of the cases where b_0 only satisfies the bending condition. Very roughly, we have bending in the zone where $b_0 \geq \underline{b}$. Condition **1)** implies that trajectories “often” meet this zone. In the other hand, Condition **2)** allows to be sure that the bending effect is not too much affected in the zone where b_0 can be negative, which could be annihilated otherwise.

With these ingredients in hand, it is quite straightforward to build a relevant reference solution (\bar{f}, \bar{E}) for the Vlasov-Poisson system with external magnetic field B_0 , for some source supported in ω . Such a solution can be constructed so that the two first moments (charge and current) are independent of c .

3.3. The reference solution in the Vlasov-Maxwell case

We call (\tilde{E}, \tilde{B}) the solutions to the Maxwell equations with sources given by $(\bar{\rho}, \bar{j} := \int \bar{f} dv, \int \bar{f} \hat{v} dv)$. Clearly this is a solution to the Vlasov-Maxwell system with a source in ω . We would like to apply Lemma 3.2 to show that this is a relevant solution as c gets large. Unfortunately, the local conservation of charge and zero mean current condition are for the moment not satisfied by \bar{f} . The idea is to add a correction g to \bar{f} which does not modify the local density of charge $\bar{\rho}$ but only modifies the current, so that all conditions are satisfied. To this end, using the fourth condition in Proposition 3.3 is crucial. The new reference solution is still relevant since in the Poisson case, the electric field is only affected by the local density of charge which has remained unchanged.

3.4. Scaling invariance of the Vlasov-Maxwell system

To conclude, we mention that using some scaling invariance of the relativistic Vlasov-Maxwell equation, it is possible to deduce another result where the speed of light is fixed (say $c = 1$) and in which the condition (3.4) is replaced by a more conventional smallness condition. We refer to [10] for the statement and proof of such a result.

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