ON THE LINEARIZED VLASOV-POISSON SYSTEM ON THE WHOLE SPACE AROUND STABLE HOMOGENEOUS EQUILIBRIA

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Abstract. We study the linearized Vlasov-Poisson system around suitably stable homogeneous equilibria on $\mathbb{R}^d \times \mathbb{R}^d$ (for any $d \geq 1$) and establish dispersive L^∞ decay estimates in the physical space.

1. Introduction

This work is concerned with the Vlasov-Poisson system on $\mathbb{R}^d \times \mathbb{R}^d$ for $d \geq 1$:

$$\begin{cases} \partial_t \mathrm{f} + v \cdot \nabla_x \mathrm{f} + \mathrm{E} \cdot \nabla_v \mathrm{f} = 0, \quad (x,v) \in \mathbb{R}^d \times \mathbb{R}^d, \\ \mathrm{E} = \nabla_x \Delta_x^{-1} (\rho - 1), \qquad \rho(t,x) = \displaystyle\int_{\mathbb{R}^d} \mathrm{f}(t,x,v)\, dv, \\ \mathrm{f}|_{t=0} = \mathrm{f}_0, \end{cases} \tag{1.1}$$

where f (resp. E) describes the distribution function of negatively charged particles (resp. the electric field) in a plasma with a fixed uniform background of ions. We are interested in the long time behavior of the solutions to (1.1) around *homogeneous* equilibria, i.e. non-negative distribution functions $\mu(v)$ satisfying

$$\int_{\mathbb{R}^d} \mu(v)\, dv = 1. \tag{1.2}$$

To this end, we consider solutions of the form $\mathrm{f}(t,x,v) = \mu(v) + f(t,x,v)$ and specifically focus on the *linearized* equations:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v \mu = 0, \quad (x,v) \in \mathbb{R}^d \times \mathbb{R}^d, \\ E = \nabla_x \Delta_x^{-1} \rho, \qquad \rho(t,x) = \displaystyle\int_{\mathbb{R}^d} f(t,x,v)\, dv, \\ f|_{t=0} = f_0. \end{cases} \tag{1.3}$$

Our goal is to establish decay in time for the density ρ of the solution to (1.3). To this purpose, we will require that μ satisfies some appropriate conditions of stability. This problem can be seen as a first step towards the understanding of relaxation properties around stable homogeneous equilibria (i.e. *Landau Damping*) for the full Vlasov-Poisson system (1.1) on the whole space.

Landau Damping was studied in the breakthrough paper [15] by Mouhot and Villani in the case of $\mathbb{T}^d \times \mathbb{R}^d$ (see also [3] and very recently [12]). All these works are based on a linear mechanism called *phase mixing*, which is specific to the free transport operator $\partial_t + v \cdot \nabla_x$ on the torus; furthermore they require perturbations of Gevrey or analytic regularity to handle the non-linear problem, in order to avoid resonances referred to as *plasma echoes*. For what concerns the whole space, an important contribution is due to Bedrossian, Masmoudi and Mouhot who considered in [4] the *screened* Vlasov-Poisson system, which corresponds to a low frequency (or equivalently, long range) regularization of the Coulomb potential, resulting in the equation

$$E = \nabla_x (1 - \Delta_x)^{-1} \rho$$

for the electric field. They relied on *dispersive* properties of the free transport operator $\partial_t + v\cdot\nabla_x$ on the whole space in the Fourier side to prove decay in finite regularity for the full non-linear system in dimensions $d \geq 3$ (with a strategy inspired by [15, 3]). In [13] we have very recently revisited this problem with another approach, namely by developing dispersive L^∞ linearized estimates in the physical space, which allowed us to use a Lagrangian strategy in the spirit of [2] for the non-linear problem (see also [16]). In particular, [13] shows that in the screened case, in all dimensions, the linear decay in the physical space is the same as for free transport, up to a logarithmic correction.

One expects the situation to be radically different for the unscreened Coulomb case (1.3), as evidenced in the pioneering works by Glassey and Schaeffer [10, 11]. In particular [10, 11] prove that in dimension $d = 1$, when μ is a Maxwellian, the L^2 norm of the density of the solution to (1.3) cannot in general decay faster than $1/(\log t)^{13/2}$ (whereas for free transport it decays like $1/t^{1/2}$). Furthermore, [10, 11] provide decay estimates, highlighting the influence of the rate of decay of μ at infinity:

- when μ is a Maxwellian, ρ decays logarithmically fast in L^2 and L^∞ norm.
- When μ decays at most polynomially fast, ρ decays polynomially fast in L^2 and L^∞ norm (with a rate that cannot be better than $1/2$ and gets worse when μ decays faster).
- On the other hand when μ is compactly supported, they show that the L^2 norm of the density may not decay at all.

In this work, we shall consider a general class of analytic homogeneous equilibria (that includes Maxwellian and power laws for example). Quantitatively, we assume that there exist $R_0 > 0$ and $C_0 > 0$ so that for all polynomials P of degree less than or equal to α_d, with $\alpha_d := d + 8$,

$$|\mathcal{F}_v(\mu)(\xi)| + |\mathcal{F}_v(P(v)\nabla_v\mu)(\xi)| \leq C_0 e^{-R_0|\xi|}, \quad \forall \xi \in \mathbb{R}^d. \tag{1.4}$$

where we use $\mathcal{F}_v$ to denote the Fourier transform.

Following [15] and [4], one could expect that a relevant notion of stability is the one in the sense of Penrose, that would correspond to asking that there is $\kappa > 0$ such that

$$\inf_{\gamma\geq 0,\, \tau\in\mathbb{R},\, \xi\in\mathbb{R}^d} \left| 1 - \int_0^{+\infty} e^{-(\gamma+i\tau)s} \frac{i\xi}{|\xi|^2}\cdot \mathcal{F}_v(\nabla_v\mu)(\xi s)\, ds \right| \geq \kappa. \tag{1.5}$$

However, as we will soon observe, though relevant in the torus case, this condition can *never* be satisfied on the whole space. This is because of a *low frequency* (in space) singularity (i.e. for small values of $|\xi|$), which is the reason why the decay that can be obtained in the screened case should not be expected here. This explains (most of) the results of [10, 11]: their strategy is based on a cut-off argument around the singularity, which accounts for why the rate of decay of μ matters in their result. We shall show that despite this singularity, with a relevant notion of stability, a natural and much stronger decay estimate that depends only on the dimension can be obtained.

A simplified version of our main result is stated in the following theorem.

Theorem 1.1. *Let $d \geq 1$. Let μ be a non-negative radial equilibrium satisfying* (1.2) *and* (1.4)*, of the form $\mu(v) = F\left(\frac{|v|^2}{2}\right)$, with $F'(s) < 0$, $\forall s \geq 0$. Consider the density $\rho(t,x)$ of the solution of* (1.3)*. Then we can decompose*

$$\rho(t,x) = \rho^R(t,x) + \rho^S_+(t,x) + \rho^S_-(t,x),$$

where for all $t \geq 2$, we have

$$\|\rho^R(t)\|_{L^\infty} \lesssim \frac{\log t}{t^d}\left(\|f_0\|_{L^1_{x,v}} + \|f_0\|_{L^1_x L^\infty_v}\right)$$

and for $k = 0, 1$,

$$\|\rho^S_\pm(t)\|_{L^\infty} \lesssim \frac{\log t}{t^{\frac{d}{2}+k-1}} \sum_{0\leq l\leq k}\left(\|\langle v\rangle^l \nabla_v^l f_0\|_{L^1_{x,v}} + \|\langle v\rangle^l \nabla_v^l f_0\|_{L^1_x L^\infty_v}\right).$$

Remark 1.1. *All the assumptions, in particular, $F' < 0$, are satisfied when μ is a Maxwellian equilibrium or a power law $\mu(v) = c_d \frac{1}{(1+|v|^2)^m}$, with m sufficiently large.*

Remark 1.2. *The rate of decay of μ does not play any role in this result.*

Remark 1.3. *Observe that we do not state any decay in L^2 and therefore this is not in contradiction with* [10, 11].

Remark 1.4. *Contrary to the screened case, we do not have $\nabla_x^k \rho$ to decay faster than ρ; this is due to the singular part $\rho_\pm^S$.*

In Theorem 1.1 (and in the more general version Theorem 2.1 below), we have only stated L^∞ type dispersive estimates. Nevertheless, we shall provide a much more precise description of the structure of ρ^R and $\rho_\pm^S$ in the following. By using by now standard interpolation estimates, we could deduce from them Strichartz estimates for example ([7], [9]). We observe that the regular part ρ^R enjoys the same decay estimates as the solution of the linearized screened Vlasov-Poisson system obtained in [13] which are themselves similar to the ones of the free transport up to the logarithmic factor. The singular part $\rho_\pm^S$ is precisely due to a singularity at the frequency $\xi = 0$, $\tau = \pm 1$ in the dispersion relation. It can be seen as the solution to a dispersive partial differential equation. Indeed, we shall show that $\rho_\pm^S$ is under the form

$$\rho_\pm^S(t,x) = \int_0^t G_\pm^S(t-s) *_x S(s)\, ds, \quad S(t) = \int_{\mathbb{R}^d} f_0(x - vt, v)\, dv$$

the kernel $G_\pm^S$ being under the form

$$G_+^S(t,x) = \int_{\mathbb{R}^d} e^{Z_\pm(\xi)t + ix\cdot\xi} A_\pm(\xi)\, d\xi$$

where $A_\pm$ is a smooth amplitude that is compactly supported for small $|\xi|$. The phase $Z_\pm(\xi)$ is such that $\mathrm{Re}\, Z_\pm(\xi) \leq 0$, in addition, $\xi \mapsto \mathrm{Re}\, Z_\pm(\xi)$ vanishes at $\xi = 0$ and is very flat (and gets flatter when μ decays faster) so that only a very weak decay connected to the rate of decay of μ can be obtained from this piece of information. This accounts for the decay results of [10, 11]. Here we shall use that the imaginary part of the phase is non-degenerate so that the decay rate $t^{-\frac{d}{2}}$ can be obtained from a stationary phase analysis. A significant part of the analysis of the paper will be to perform a careful analysis of the singularity of the dispersion relation at $\tau = \pm 1$, $\xi = 0$ and to justify that it gives rise to the above singular term.

2. Statement of the theorem with general assumptions on the equilibrium

As a matter of fact, Theorem 1.1 is a special case of a more general result, allowing for a wider class of homogeneous equilibria (not necessarily radial) that satisfy a series of assumptions, which we now present.

Symmetry assumptions. For all monomials P of odd degree $k \leq \alpha_d - 1$, we require that

$$\int_{\mathbb{R}^d} P(v)\mu(v)\, dv = 0. \tag{2.1}$$

We shall also ask that for all $p \in \mathbb{N} \setminus \{0\}$ such that $2p \leq \alpha_d - 1$,

$$\exists C_\mu^p, \quad \forall \xi \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d} (\xi \cdot v)^{2p} \mu(v)\, dv = C_\mu^p |\xi|^{2p}. \tag{2.2}$$

Observe that (2.1) is in particular satisfied when μ is even and (2.2) when μ is radial; however both can also be satisfied assuming (many) algebraic identities on integrals of μ against polynomials. For most of the arguments, we shall only need (2.1) and we will emphasize precisely where the additional assumption (2.2) is needed in the paper.

Stability assumptions. Two stability assumptions are required.
Assumption (H1). We shall first ask for the stability condition: for every $\xi \neq 0$

$$\inf_{\gamma\geq 0,\, \tau\in\mathbb{R}} \left|1 - \int_0^{+\infty} e^{-(\gamma+i\tau)s}\, \frac{i\xi}{|\xi|^2}\cdot \mathcal{F}_v(\nabla_v \mu)(\xi s)\, ds\right| > 0, \tag{2.3}$$

which is a weaker non-quantitative version of (1.5)

In order to tame the effect of the singularity at $\xi = 0$, we shall require *another Penrose stability* condition. To this end, let us introduce

$$m_{KE}(z,\eta) = -\int_0^{+\infty} e^{-(\gamma+i\tau)s}\, \frac{i\eta}{|\eta|^2}\cdot \sum_{k,l} \eta_k\eta_l \mathcal{F}_v(v_k v_l \nabla_v\mu)(\eta s)\, ds, \quad z = \gamma + i\tau.$$

For $\eta \in \mathbb{S}^{d-1}$, thanks to (1.4), we observe that m_{KE} is holomophic in $\mathrm{Re}\, z > -R_0$.
Assumption (H2). For every $\eta \in \mathbb{S}^{d-1}$, there is only one zero of $z \mapsto 1 - m_{KE}(z,\eta)$ on $\mathrm{Re}\, z = 0$ which is $z = 0$. Moreover it verifies

$$\partial_z m_{KE}(0,\eta) = 0, \quad \partial_z^2 m_{KE}(0,\eta) \neq 0, \quad \forall \eta \in \mathbb{S}^{d-1}. \tag{2.4}$$

This condition can be interpreted as a kind of Penrose stability condition for the so-called *kinetic Euler* equation, which is a singular Vlasov equation arising in the quasineutral limit of the Vlasov-Poisson system and in Brenier's incompressible optimal transport [5], [6].

We are finally in position to state the main result of the paper.

Theorem 2.1. *Assume that* (1.2), (1.4), (2.1), (2.2), *(H1) and (H2) are satisfied. Then the conclusions of Theorem 1.1 hold.*

Remark 2.1. *The assumption (H2) can be replaced by*

Assumption (H2'). *For every $\eta \in \mathbb{S}^{d-1}$, there is no zero of $z \mapsto 1 - m_{KE}(z,\eta)$ on* $\mathrm{Re}\, z = 0$.

The proof of Theorem 2.1 gets slightly simplified in that case. However we have decided to focus on (H2) as (H2') is never satisfied for radial equilibria.

Theorem 1.1 follows from Theorem 2.1 once that we have checked that the radial equilibria $\mu = F(|v|^2/2)$ that we consider satisfy all required assumptions:

- we have already seen that (2.1) and (2.2) are satisfied when μ is radial.
- As seen from [15, Proposition 2.1 and Remark 2.2], the assumption (H1) is verified for radial equilibria in any dimension assuming that $F' < 0$.
- Finally the assumption (H2) is also satisfied if $F' < 0$. We postpone the proof of this result to an appendix, see Section 8.

The rest of the paper is dedicated to the proof of Theorem 2.1.

3. Reduction to kernel estimates

We study the linear equation

$$\rho(t,x) = \int_0^t \int_{\mathbb{R}^d} -[\nabla_x \Delta_x^{-1}\rho](s, x-(t-s)v)\cdot \nabla_v\mu(v)\, dvds + S(t,x), \quad t\geq 0, \tag{3.1}$$

with S being a given source term (as we shall see later, we can rewrite (1.3) under this form by integrating along the characteristics of the free transport). In what follows, we extend ρ and S by zero for $t < 0$ so that the equation (3.1) is satisfied for $t \in \mathbb{R}$. For $\gamma > 0$ sufficiently large, by using the Fourier transform in space and time, we get that the solution of (3.1) is given by

$$\mathcal{F}(e^{-\gamma t}\rho)(\tau,\xi) = m_{VP}(\gamma,\tau,\xi)\mathcal{F}(e^{-\gamma t}S)(\tau,\xi),$$

where $\mathcal{F}$ denotes the Fourier transform in time and space, and hence that

$$\rho(t,x) = S + \left(e^{\gamma t}\left(\mathcal{F}^{-1}\frac{m_{VP}(\gamma,\cdot)}{1-m_{VP}(\gamma,\cdot)}\right)\right) *_{t,x} S = S + G *_{t,x} S, \tag{3.2}$$

where

$$G(t,x) = \int_{\mathbb{R}\times\mathbb{R}^d} e^{\gamma t + i\tau t} e^{ix\cdot\xi}\frac{m_{VP}(\gamma,\tau,\xi)}{1-m_{VP}(\gamma,\tau,\xi)}\, d\tau d\xi. \tag{3.3}$$

The aim of the remaining will be to estimate the kernel G. Note that the definition of the kernel G depends on γ, but that in regions where the integrand is an holomorphic function of $z=\gamma+i\tau$ it actually does not depend on γ since we can appropriately change the integration contour, via the Cauchy formula, without changing G. In particular, by taking the limit $\gamma\to+\infty$, we get that $G_{|t<0}=0$.

Precisely, in this paper, we shall prove:

Theorem 3.1. *For $t\le 1$, we have the estimate*

$$\|G(t)\|_{L^\infty} \lesssim \frac{1}{t^{d-1}}, \quad \|G(t)\|_{L^1} \lesssim t.$$

Moreover, assuming (1.2), (1.4), (2.2), *(H1), (H2), we can write for $t\ge 1$,*

$$G = G^R(t,x) + G^S_+(t,x) + G^S_-(t,x),$$

where

$$\|G^R(t)\|_{L^\infty} \lesssim \frac{1}{t^{d+1}}, \quad \|G^R(t)\|_{L^1} \lesssim \frac{1}{t}, \quad \forall t\ge 1$$

and

$$\|G^S_\pm(t)\|_{L^\infty} \lesssim \frac{1}{t^{\frac{d}{2}}}, \quad \|G^S_\pm(t)\|_{L^2} \lesssim 1, \quad \forall t\ge 1.$$

A more accurate description of $G^S_\pm$ is given in Proposition 5.4. They can be seen as the kernel of the propagator of a dispersive PDE.

4. Properties of the symbol m_{VP}

Let us set for $(\gamma,\tau,\xi)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d$,

$$m_{VP}(\gamma,\tau,\xi) = \int_0^{+\infty} e^{-(\gamma+i\tau)s}\,\frac{i\xi}{|\xi|^2}\cdot\mathcal{F}_v(\nabla_v\mu)(\xi s)\, ds, \tag{4.1}$$

$$m_{VB}(\gamma,\tau,\xi) = \int_0^{+\infty} e^{-(\gamma+i\tau)s}\, i\xi\cdot\mathcal{F}_v(\nabla_v\mu)(\xi s)\, ds, \tag{4.2}$$

$$m_{KE}(\gamma,\tau,\xi) = -\int_0^{+\infty} e^{-(\gamma+i\tau)s}\,\frac{i\xi}{|\xi|^2}\cdot\sum_{k,l}\xi_k\xi_l\mathcal{F}_v(v_kv_l\nabla_v\mu)(\xi s)\, ds. \tag{4.3}$$

Remark 4.1. *As already mentioned, m_{KE} is the symbol associated to the Kinetic Euler equation. It turns out that the symbol m_{VB} is the one associated to the so-called* Vlasov-Benney *equation (see e.g.* [1]*) which is another singular Vlasov equation that shows up in the quasineutral limit of the Vlasov-Poisson system* [14]*.*

Let us define $\Omega_{R_0} = \mathcal{A}\cap\mathcal{C}_{R_0}$ where,

$$\mathcal{A} = \left\{(\gamma,\tau,\xi)\in\mathbb{R}^{d+2},\ \frac{1}{2} < |\gamma|+|\tau|+|\xi| < 2\right\}, \quad \mathcal{C}_{R_0} = \left\{(\gamma,\tau,\xi)\in\mathbb{R}^{d+2},\ \xi\neq 0,\ \gamma > -R_0|\xi|\right\}.$$

4.1. **Estimates of m_{VP}.** The following is an adaptation of Lemma 2.2 in [13]. We get stronger properties due to the regularity assumption (1.4) and the symmetry assumption (2.1).

Proposition 4.1. *Assuming* (1.2), (1.4) *and* (2.1), *we have the following properties. For every* $(\gamma, \tau, \xi) \in \mathcal{C}_{R_0}$,

$$m_{VP}(\gamma,\tau,\xi) = \frac{1}{|\xi|^2} m_{VB}(\gamma,\tau,\xi), \quad m_{VP}(\gamma,\tau,\xi) = -\frac{1}{(\gamma+i\tau)^2}(1 - m_{KE}(\gamma,\tau,\xi)) \tag{4.4}$$

The symbols $m_{VB}(\gamma,\tau,\xi)$ and $m_{KE}(\gamma,\tau,\xi)$ are for $\xi \neq 0$ holomorphic with respect to the variable $z = \gamma + i\tau$ in $\gamma > -R_0|\xi|$. Moreover, they are positively homogeneous of degree zero and m_{VB}, $m_{KE} \in \mathscr{C}^{\alpha_d - 2}(\mathcal{C}_{R_0/2})$. Quantitatively, there exists $C > 0$ such that

$$|\partial_z^\alpha \partial_\xi^\beta m_{VB}(\gamma,\tau,\xi)| + |\partial_z^\alpha \partial_\xi^\beta m_{KE}(\gamma,\tau,\xi)| \leq \frac{C}{|(\gamma,\tau,\xi)|^{|\alpha|+|\beta|}}, \ \forall |\alpha| + |\beta| \leq \alpha_d - 2, \ \forall (\gamma,\tau,\xi) \in \mathcal{C}_{R_0/2}, \tag{4.5}$$

where we use $\partial_z = \partial_\gamma - i\partial_\tau$.

Remark 4.2. *In the following, we shall often abuse notations and write the symbols as functions of (γ, τ, ξ) or (z, ξ) depending on what is most convenient.*

Proof. Let us first observe that thanks to (1.4), m_{VP}, m_{KE} and m_{VB} are well-defined in $\mathcal{C}_{R_0}$ and holomorphic in z for $\xi \neq 0$ and $\mathrm{Re}\, z > -R_0|\xi|$. Let us prove (4.4). The first relation is trivial. For the second one, by two successive integrations by parts in s, we obtain

$$\begin{aligned} m_{VP}(\gamma,\tau,\xi) &= \frac{1}{\gamma+i\tau}\int_0^{+\infty} e^{-(\gamma+i\tau)s}\, \frac{i\xi}{|\xi|^2}\cdot \partial_s \left(\mathcal{F}_v(\nabla_v \mu)(\xi s)\right) ds \\ &= \frac{1}{(\gamma+i\tau)^2}\int_0^{+\infty} e^{-(\gamma+i\tau)s}\, \frac{i\xi}{|\xi|^2}\cdot \partial_s^2 \left(\mathcal{F}_v(\nabla_v \mu)(\xi s)\right) ds \\ &\qquad + \frac{1}{(\gamma+i\tau)^2}\frac{i\xi}{|\xi|^2}\cdot \partial_s \left(\mathcal{F}_v(\nabla_v \mu)(\xi s)\right)|_{s=0}. \end{aligned}$$

Since by (1.2),

$$\partial_s^2 \left(\mathcal{F}_v(\nabla_v \mu)(\xi s)\right) = -\sum_{k,l} \xi_k \xi_l \mathcal{F}_v(v_k v_l \nabla_v \mu)(\xi s), \quad \frac{i\xi}{|\xi|^2}\cdot \partial_s \left(\mathcal{F}_v(\nabla_v \mu)(\xi s)\right)|_{s=0} = -1,$$

we finally get (4.4). The degree zero homogeneity property comes from a straightforward change of variable. It remains to prove (4.5). We first give the proof for m_{VB}.

Since we have

$$m_{VB}(\gamma,\tau,\xi) = \int_0^{+\infty} e^{-(\gamma+i\tau)t} i\xi \cdot \widehat{\nabla_v \mu}(t\xi)\, dt,$$

we get by using (1.4), that

$$|m_{VB}(\gamma,\tau,\xi)| \leq C\int_0^{+\infty} |\xi| e^{-R_0|\xi|t/2}\, dt \leq C, \quad \forall (\gamma,\tau,\xi) \in \mathcal{C}_{R_0/2}.$$

We now estimate the derivatives in $\Omega_{R_0/2}$; let us first handle the case when $|\xi| \geq \frac{1}{4}$. Thanks to (1.4), we also have that

$$|\partial_z^\alpha \partial_\xi^\beta m_{VB}(\gamma,\tau,\xi)| \lesssim \int_0^{+\infty} \langle t\rangle^{|\alpha|+|\beta|} |\xi| e^{-R_0|\xi|t/2}\, dt$$

and therefore, for $|\xi| \geq \frac{1}{4}$ and $|\alpha| + |\beta| \leq \alpha_d$, we obtain

$$|\partial_z^\alpha \partial_\xi^\beta m_{VB}(\gamma,\tau,\xi)| \lesssim 1, \qquad (\gamma,\tau,\xi) \in \Omega_{R_0/2}, \quad |\xi| \geq \frac{1}{4}. \tag{4.6}$$

Let us next consider the case $|\xi| \le \frac{1}{4}$, in which we make use of the fact that $|z|$ is positively bounded from below, recalling $(\gamma,\tau,\xi) \in \mathcal{A}$. Integrating by parts again, we get for every $n = 2, \cdots, \alpha_d$,

$$m_{VB}(\gamma,\tau,\xi) = \sum_{k=2}^{n} \frac{1}{z^k}\mathscr{P}_k(\xi) + \frac{1}{z^n} R_n(\gamma,\tau,\xi) \tag{4.7}$$

where

$$\mathscr{P}_k(\xi) = (-1)^{k-1} i^k \xi \cdot \mathcal{F}\left(v^{\otimes k-1}\nabla_v \mu\right)(0) : \xi^{\otimes k-1},$$

$$R_n(\gamma,\tau,\xi) = \int_0^{+\infty} e^{-(\gamma+i\tau)t} r_n(t,\xi)\, dt, \quad r_n(t,\xi) = (-1)^n i^n \xi \cdot \mathcal{F}\left(v^{\otimes n}\nabla_v\mu\right)(t\xi) : \xi^{\otimes n},$$

with the definition

$$\xi \cdot \mathcal{F}(v^{\otimes k}\nabla_v \mu)(\zeta) : \xi^{\otimes k} = \sum_{j_0, j_1, \cdots j_k} \xi_{j_0}\xi_{j_1}\cdots\xi_{j_k} \mathcal{F}(v_{j_1}\cdots v_{j_k}\partial_{v_{j_0}}\mu)(\zeta).$$

Note that $\mathscr{P}_k$ is a homogeneous polynomial of degree k. Thanks to (1.4), we have

$$|r_n(t,\xi)| \lesssim |\xi|^{n+1} e^{-R_0|\xi|t}.$$

More generally, we have for all $|\beta| \le n$,

$$|\partial_\xi^\beta r_n(t,\xi)| \le |\xi|^{n+1-|\beta|} e^{-R_0|\xi|t}.$$

Consequently, applying derivatives to the expansion (4.7) and using the above estimates with $n = \alpha_d$, we get for $|\xi| \le \frac{1}{4}$ and $(\gamma,\tau,\xi) \in \Omega_{R_0/2}$ (which in particular implies that $|z| \ge \frac{1}{4}$),

$$|\partial_z^\alpha \partial_\xi^\beta m_{VB}(\gamma,\tau,\xi)| \lesssim 1 + \int_0^{+\infty} t^{|\alpha|} |\xi|^{\alpha_d+1-|\beta|} e^{-R_0|\xi|t/2}\, dt \lesssim 1 + \int_0^{+\infty} s^{|\alpha|}|\xi|^{\alpha_d+1-|\beta|-|\alpha|} e^{-R_0 s/2}\, ds.$$

Thus, we get for $|\alpha| + |\beta| \le \alpha_d$,

$$|\partial_z^\alpha \partial_\xi^\beta m_{VB}(\gamma,\tau,\xi)| \lesssim 1, \qquad (\gamma,\tau,\xi) \in \Omega_{R_0/2}, \quad |\xi| \le \frac{1}{4}.$$

This, together with (4.6), concludes the proof of the estimates for m_{VB} on $\Omega_{R_0/2}$; we finally obtain (4.5) for m_{VB} by degree zero homogeneity.

Let us now prove the estimates for m_{KE} on $\Omega_{R_0/2}$. The same argument as above applies for $|\xi| \ge 1/4$. Thus it suffices to study $|\xi| \le 1/4$. As before, we can integrate by parts to get that

$$m_{KE}(z,\xi) = \sum_{k=2}^{n} \frac{1}{z^k}\mathcal{Q}_k(\xi) + \frac{1}{z^n} R_n^{KE}(\gamma,\tau,\xi) \tag{4.8}$$

where

$$\mathcal{Q}_k(\xi) = (-1)^k i^k \frac{\xi}{|\xi|^2} \cdot \mathcal{F}\left(v^{\otimes k+1}\nabla_v\mu\right)(0) : \xi^{\otimes k+1},$$

$$R_n^{KE}(\gamma,\tau,\xi) = \int_0^{+\infty} e^{-(\gamma+i\tau)t} r_n^{KE}(t,\xi)\, dt, \quad r_n^{KE}(t,\xi) = (-1)^{n+1} i^n \frac{\xi}{|\xi|^2} \cdot \mathcal{F}\left(v^{\otimes n+2}\nabla_v\mu\right)(t\xi) : \xi^{\otimes n+2}.$$

In this case, we need to study more carefully the structure of this expansion since the function $\xi \otimes \xi/|\xi|^2$ multiplied by a polynomial of ξ is not necessarily a smooth function of ξ. We first observe that if k is odd, we have

$$\mathcal{Q}_k(\xi) = 0, \quad \forall \xi.$$

Indeed, we can integrate by parts and use that

$$\int_{\mathbb{R}^d} v^{\otimes k} \mu(v)\, dv = 0$$

if k is odd thanks to the symmetry assumption (2.1). We thus have the expansion

$$m_{KE}(z,\xi) = \sum_{p=1}^{l} \frac{1}{z^{2p}} \mathcal{Q}_{2p}(\xi) + \frac{1}{z^{2l+1}} R^{KE}_{2l+1}(\gamma,\tau,\xi). \tag{4.9}$$

Moreover, we have

$$\begin{aligned}\mathcal{Q}_{2p}(\xi) = (-1)^p \frac{\xi}{|\xi|^2} \cdot \mathcal{F}\left(v^{\otimes 2p+1} \nabla_v \mu\right)(0) : \xi^{\otimes 2p+1} \\ = (-1)^p \frac{1}{|\xi|^2} \sum_{j_0,\cdots,j_{2p+1}} \xi_{j_0} \cdots \xi_{j_{2p+1}} \int_{\mathbb{R}^d} v_{j_1} \cdots v_{j_{2p+1}} \partial_{v_{j_0}} \mu(v)\, dv.\end{aligned}$$

Integrating by parts, we observe that if $j_1, \cdots, j_{2p+1}$ are all different from j_0, the integral vanishes, therefore, after relabelling, we get that

$$\begin{aligned}\mathcal{Q}_{2p}(\xi) = (-1)^{p+1}(2p+1) \frac{1}{|\xi|^2} \sum_{j_0,j_2\cdots,j_{2p+1}} (\xi_{j_0})^2 \xi_{j_2} \cdots \xi_{j_{2p+1}} \int_{\mathbb{R}^d} v_{j_2} \cdots v_{j_{2p+1}} \mu(v)\, dv \\ = (-1)^{p+1}(2p+1) \sum_{j_2,\cdots,j_{2p+1}} \xi_{j_2} \cdots \xi_{j_{2p+1}} \int_{\mathbb{R}^d} v_{j_2} \cdots v_{j_{2p+1}} \mu(v)\, dv \\ = (-1)^{p+1}(2p+1) \int_{\mathbb{R}^d} (\xi \cdot v)^{2p} \mu(v)\, dv.\end{aligned}$$

We have thus obtained that $\mathcal{Q}_{2p}$ is a polynomial in ξ and hence a smooth function of ξ. We can then get from the expansion (4.9) and similar estimates as for m_{VB} that the derivatives with respect to z and ξ of order less than $\alpha_d - 2$ are uniformly bounded for $|\xi| \leq 1/4$. Finally, the estimate (4.5) follows by using the degree zero homogeneity. □

In the proof of Proposition 4.1, we have obtained a refined asymptotic expansion of m_{KE}, for which we have relied on the symmetry assumption (2.1). We gather this very useful statement in the following lemma.

Lemma 4.1. *Assuming* (1.2), (1.4) *and* (2.1), *we have the following expansion of* m_{KE} *for all* $l \leq \lfloor \frac{\alpha_d - 3}{2} \rfloor$*:*

(4.10)
$$m_{KE}(z,\xi) = \sum_{p=1}^{l} \frac{1}{z^{2p}} \mathcal{Q}_{2p}(\xi) + \frac{1}{z^{2l+1}} R^{KE}_{2l+1}(\gamma,\tau,\xi),$$

(4.11)
$$\mathcal{Q}_{2p}(\xi) = (-1)^{p+1}(2p+1) \xi^{\otimes 2p} : \int_{\mathbb{R}^d} v^{\otimes 2p} \mu(v)\, dv = (-1)^{p+1}(2p+1) \int_{\mathbb{R}^d} (\xi \cdot v)^{2p} \mu(v)\, dv,$$

(4.12)
$$R^{KE}_{2l+1}(z,\xi) = \int_0^{+\infty} e^{-(\gamma+i\tau)t} r^{KE}_{2l+1}(t,\xi)\, dt,\ r^{KE}_{2l+1}(t,\xi) = (-1)^{l+1} \frac{\xi}{|\xi|^2} \cdot \mathcal{F}\left(v^{\otimes 2l+2} \nabla_v \mu\right)(t\xi) : \xi^{\otimes 2l+3}$$

where the remainder satisfies uniformly for $(\gamma,\tau,\xi) \in \mathcal{C}_{R_0/2}$ *the estimate*

$$|\partial_z^\alpha \partial_\xi^\beta R^{KE}_{2l+1}(z,\xi)| \lesssim |\xi|^{2l+1-|\alpha|-|\beta|}, \quad |\alpha| + |\beta| \leq 2l+1. \tag{4.13}$$

In particular, we get

$$\mathcal{Q}_2(\xi) = -3 H_\mu \xi \cdot \xi, \quad H_\mu = \int_{\mathbb{R}^d} v \otimes v\, \mu(v)\, dv.$$

Under the additional symmetry assumption (2.2), *we have*

$$\mathcal{Q}_{2p}(\xi) = (-1)^p(2p+1)|\xi|^{2p}C_\mu^p \tag{4.14}$$

where

$$C_\mu^p = \int_{\mathbb{R}^d} v_1^{2p}\mu(v)\,dv.$$

In particular, we have

$$C_\mu := C_\mu^1 = \frac{1}{d}\int_{\mathbb{R}^d} |v|^2\,\mu(v)\,dv.$$

As a consequence of Proposition 4.1, we obtain estimates for m_{VP}.

Corollary 4.1. *The symbol $m_{VP}(\gamma,\tau,\xi)$ for $\xi \neq 0$ is holomorphic with respect to the variable $z = \gamma + i\tau$ in $\operatorname{Re} z > -R_0|\xi|$. Moreover, it is positively homogeneous of degree -2 and $m_{VP} \in \mathscr{C}^{\alpha_d-2}(\mathcal{C}_{R_0/2})$. Quantitatively, there exists $C > 0$ such that*

$$|\partial_z^\alpha \partial_\xi^\beta m_{VP}(\gamma,\tau,\xi)| \leq \frac{C}{|(\gamma,\tau,\xi)|^{2+|\alpha|+|\beta|}},\ \forall |\alpha| + |\beta| \leq \alpha_d - 2,\ \forall (\gamma,\tau,\xi) \in \mathcal{C}_{R_0/2}. \tag{4.15}$$

Proof. The fact that m_{VP} is positively homogeneous of degree -2 follows from a change of variables. For the other properties, by using the previous lemma, it suffices to observe that

$$m_{VP}(\gamma,\tau,\xi) = \frac{1}{8|\xi|^2 - (\gamma + i\tau)^2}\left(8m_{VB}(\gamma,\tau,\xi) + (1 - m_{KE}(\gamma,\tau,\xi)\right). \tag{4.16}$$

We get that the modulus of the denominator for $(\gamma,\tau,\xi) \in \Omega_{R_0/2}$ is positively uniformly bounded from below. Indeed, for $|\xi|^2 \geq 1/4$, we observe that

$$\left|8|\xi|^2 - (\gamma + i\tau)^2\right| \geq 8|\xi|^2 - \gamma^2 > c_0,$$

for some $c_0 > 0$. Otherwise, if $|\xi|^2 < 1/4$, we must have $\tau^2 + \gamma^2 > 1/4$, in which case

$$\left|8|\xi|^2 - (\gamma + i\tau)^2\right| \geq |\tau^2 - \gamma^2| + 2|\tau||\gamma| > c_1,$$

for some $c_1 > 0$. Consequently, we can apply (4.5) and the estimates (4.15) follow since m_{VP} is positively homogeneous of degree -2. $\square$

4.2. **Zeroes of $1 - m_{VP}$.** In this section we give a sharp description of the zeroes of $1 - m_{VP}$. As we shall see, they are localized in the region $|\gamma| \leq \varepsilon_3|\xi|$, $|\tau \pm 1| \leq \varepsilon_3|\xi|$ and $|\xi| \leq \varepsilon_3$ for some small ε_3. Using the implicit function theorem, we are able to describe them by smooth curves.

Proposition 4.2. *Assuming that* (1.2), (1.4) *and* (2.1) *hold, we have the following properties:*

i) There exists $M > 0$ such that for every $(\gamma,\tau,\xi) \in \mathcal{C}_{R_0/2}$ and $|(\gamma,\tau,\xi)| \geq M$, we have

$$|1 - m_{VP}(\gamma,\tau,\xi)| \geq \frac{1}{2}.$$

ii) Assuming (H1), for every $\delta > 0$, there exists $c_\delta > 0$ and $R_\delta \in (0, R_0/2]$ such that for every $(\gamma,\tau,\xi) \in \mathcal{C}_{R_\delta}$ with $|\xi| \geq \delta$, we have

$$|1 - m_{VP}(\gamma,\tau,\xi)| \geq c_\delta.$$

iii) Assuming (H2), there exists $R_1 \in (0, R_0/2]$, $\varepsilon_1 > 0$ and $C_{\varepsilon_1} > 0$, $c_{\varepsilon_1} > 0$ such that for every $(\gamma,\tau,\xi) \in \mathcal{C}_{R_1}$ and $|\gamma| \leq \varepsilon_1|\xi|$, $|(\tau,\xi)| \leq \varepsilon_1$, we have

$$\left|\frac{m_{VP}(\gamma,\tau,\xi)}{1 - m_{VP}(\gamma,\tau,\xi)}\right| \leq C_{\varepsilon_1}, \quad |z^2 + 1 - m_{KE}(z,\xi)| \geq c_{\varepsilon_1}\min(1, |\tilde{z}|^2), \quad z = |\xi|\tilde{z}. \tag{4.17}$$

iv) There exists $\varepsilon_2 \in (0, \min(R_0/2, \varepsilon_1)]$, $A_0 \geq 1$, such that for every $\varepsilon \in (0, \varepsilon_2]$, for every $A \geq A_0$ and for every $(\gamma, \tau, \xi) \in \mathcal{C}_{R_1}$, with $|\xi| \leq \varepsilon$, $|\gamma| \leq \varepsilon|\xi|$, and $||\tau| - 1| \geq A\varepsilon|\xi|$, $\frac{1}{\varepsilon_1} \geq |\tau| \geq \varepsilon_1$, there holds

$$|1 - m_{VP}(\gamma, \tau, \xi)| \geq A\,\varepsilon\,\varepsilon_1^2|\xi|/4. \tag{4.18}$$

v) Assuming (H1), there exists $\varepsilon_3 > 0$, $\varepsilon_3 \in (0, \min(R_0/2, \varepsilon_2)]$ such that for every $\xi \neq 0$, the zeroes of $1 - m_{VP}(\gamma, \tau, \xi)$ with $|\xi| \leq \varepsilon_3$, $|\gamma| \leq \varepsilon_3|\xi|$, $||\tau| - 1| \leq \varepsilon_3|\xi|$ are given by two $\mathscr{C}^1$ curves

$$Z_\pm(r, \omega) = \pm i + r\Gamma_\pm(r, \omega) + ir\mathcal{T}_\pm(r, \omega)$$

where $\xi = r\omega$, $\omega \in \mathbb{S}^{d-1}$, $\Gamma_\pm \leq 0$, $\Gamma_\pm(0, \omega) = 0$, $\partial_r\Gamma_\pm(0, \omega) = 0$, $\Gamma_\pm(r, \omega) < 0$ for $r \neq 0$ and $\mathcal{T}_\pm$ is real with $\mathcal{T}_\pm(0, \omega) = 0$, $\partial_r\mathcal{T}_\pm(0, \omega) \neq 0$.

Proof. Let us start with i). To this end, we can apply Corollary 4.1. This entails that

$$|m_{VP}(\gamma, \tau, \xi)| \leq \frac{C}{|(\gamma, \tau, \xi)|^2}, \quad \forall(\gamma, \tau, \xi) \in \mathcal{C}_{R_0/2}$$

and hence

$$|1 - m_{VP}(\gamma, \tau, \xi)| \geq \frac{1}{2}$$

if $|(\gamma, \tau, \xi)|$ is sufficiently large.

Let us prove ii). By using i), the estimate is true if we have in addition $|(\gamma, \tau, \xi)| \geq M$, it thus suffices to consider the case that $|\xi| \geq \delta$ and $|(\gamma, \tau, \xi)| \leq M$. By (H1) and by compactness, we get that

$$|1 - m_{VP}(\gamma, \tau, \xi)| \geq 2c_\delta$$

for some $c_\delta > 0$ if $\gamma \geq 0$. By continuity, the inequality without the factor 2 remains true for $\gamma = -\alpha|\xi|$ for $\alpha \leq c$ with $c > 0$ small enough.

To prove iii), we observe that by Proposition 4.1, we can write

$$\frac{m_{VP}}{1 - m_{VP}} = -\frac{1}{z^2}\frac{1 - m_{KE}}{1 + \frac{1}{z^2}(1 - m_{KE})} = \frac{1 - m_{KE}}{z^2 + 1 - m_{KE}}. \tag{4.19}$$

By degree zero homogeneity of m_{KE}, we can set $\tilde{z} = z/|\xi|$, $\eta = \xi/|\xi|$, with $\tilde{z} = \tilde{\gamma} + i\tilde{\tau}$ and $|\tilde{\gamma}| \leq \varepsilon_1$, $|\tilde{\tau}| \leq \varepsilon_1/|\xi|$. This yields

$$\frac{m_{VP}}{1 - m_{VP}}(z, \xi) = \frac{1 - m_{KE}(\tilde{z}, \eta)}{|\xi|^2\tilde{z}^2 + 1 - m_{KE}(\tilde{z}, \eta)}.$$

By using (4.10), we have that uniformly for $|\tilde{\gamma}| \leq R_0/2$ and $\eta \in \mathbb{S}^{d-1}$,

$$\lim_{|\tilde{\tau}| \to +\infty} |m_{KE}(\tilde{z}, \eta)| = 0.$$

Therefore, for $|\tilde{\tau}| \geq M$ sufficiently large

$$|1 - m_{KE}(\tilde{z}, \eta)| \geq \frac{1}{2}$$

and hence for ε_1 sufficiently small, we get

$$\left||\xi|^2\tilde{z}^2 + 1 - m_{KE}(\tilde{z}, \eta)\right| = \left|z^2 + 1 - m_{KE}(\tilde{z}, \eta)\right| \geq \frac{1}{4}.$$

As a consequence, we conclude that $\left|\frac{m_{VP}}{1 - m_{VP}}\right|$ is bounded.

In a similar way, for every $\tilde{\varepsilon} > 0$ if $\tilde{\varepsilon} \leq |\tilde{\tau}| \leq M$, $1 - m_{KE}(i\tilde{\tau}, \eta)$ does not vanish thanks to (H2). By compactness and continuity this remains true uniformly for $\tilde{\gamma}$ sufficiently small and $\eta \in \mathbb{S}^{d-1}$. In this regime, we thus also get $\left||\xi|^2\tilde{z}^2 + 1 - m_{KE}(\tilde{z}, \eta)\right|$ is uniformly strictly positive and also that $\frac{m_{VP}}{1 - m_{VP}}$ is bounded.

Consequently there only remains to study the vicinity of $\tilde{z} = 0$. From (H2), we have that

$$1 - m_{KE}(\tilde{z}, \eta) = a_2(\eta)\tilde{z}^2 + O(\tilde{z}^3), \tag{4.20}$$

where by compactness, $\inf_{\mathbb{S}^{d-1}} |a_2(\eta)| \geq c_s > 0$ for some $c_s > 0$. In particular, we find that

$$|\xi|^2\tilde{z}^2 + 1 - m_{KE}(\tilde{z}, \eta) = (a_2(\eta) - |\xi|^2)\tilde{z}^2 + O(\tilde{z}^3)$$

and hence that for ε_1 and hence $|\xi|$ sufficiently small,

$$\left||\xi|^2\tilde{z}^2 + 1 - m_{KE}(\tilde{z}, \eta)\right| \geq \frac{c_s}{2}|\tilde{z}|^2.$$

This also yields that $\frac{m_{VP}}{1-m_{VP}}$ is uniformly bounded thanks to (4.19)–(4.20).

Next, we prove iv). We use again that

$$1 - m_{VP}(\gamma, \tau, \xi) = \frac{1}{(\gamma + i\tau)^2}\Big((\gamma + i\tau)^2 + (1 - m_{KE}(\gamma, \tau, \xi)\Big). \tag{4.21}$$

This yields, as $|\tau|^2 \leq 1/\varepsilon_1^2$,

$$|1 - m_{VP}(\gamma, \tau, \xi)| \geq \frac{\varepsilon_1^2}{2}\left|(\gamma + i\tau)^2 + (1 - m_{KE}(\gamma, \tau, \xi))\right|.$$

We shall need the behavior of $m_{KE}(\gamma, \tau, \xi)$ close to $\xi = 0$. By using the expansion (4.10) for $l = 1$, we obtain that in this regime, for some $C > 0$,

$$|m_{KE}(\gamma, \tau, \xi)| \leq \frac{C}{\varepsilon_1^2}|\xi|^2,$$

therefore, we obtain that for ε_2 sufficiently small

$$\begin{aligned}\left|(\gamma + i\tau)^2 + (1 - m_{KE}(\gamma, \tau, \xi))\right| &\geq |\gamma + i(\tau - 1)||\gamma + i(\tau + 1)| - |m_{KE}(\gamma, \tau, \xi)| \\ &\geq |\tau - 1|\,|\tau + 1| - \frac{C}{\varepsilon_1^2}|\xi|^2 \geq \left(\frac{2}{3}A - \frac{C}{\varepsilon_1^2}\right)\varepsilon|\xi|.\end{aligned} \tag{4.22}$$

We thus find (4.18) for ε_2 sufficiently small and $A \geq A_0$ sufficiently large.

We finally prove v). We use again (4.21), we have to study the zeroes of

$$g(z, \xi) = z^2 + 1 - m_{KE}(\gamma, \tau, \xi).$$

Writing $z = \pm i + r\mathfrak{z}$, $\xi = r\omega$, with $|\mathfrak{z}|$ small, we get by using Lemma 4.1 and the expansion (4.10) that

$$g(z, \xi) = \pm 2ir\mathfrak{z} + r^2\mathfrak{z}^2 - \frac{1}{(\pm i + r\mathfrak{z})^2}\Big(3H_\mu\omega \cdot \omega\, r^2 + r^4 m_2(\pm i + r\mathfrak{z}, r, \omega)\Big), \tag{4.23}$$

where m_2 is a smooth function of its arguments. We can thus set

$$g(z, \xi) = r f_\pm(\mathfrak{z}, r, \omega),$$

where

$$f_\pm(\mathfrak{z}, r, \omega) = \pm 2i\mathfrak{z} + r\mathfrak{z}^2 - \frac{r}{(\pm i + r\mathfrak{z})^2}\Big(3H_\mu\omega \cdot \omega + r^2 m_2(\pm i + r\mathfrak{z}, r, \omega)\Big). \tag{4.24}$$

It thus suffices to study the zeroes of $f_\pm$ for $|\mathfrak{z}|$ sufficiently small, $r > 0$ close to zero. We would like to use the implicit function theorem, nevertheless, since $r = 0$ is on the boundary of the domain of definition of f, we shall first look for a smooth extension of $f_\pm$ for small negative r. We can use again the expansion (4.10) in Lemma 4.1 to observe that $r^2 m_2$ can be expanded as a polynomial in r with even powers plus a remainder of order $\mathcal{O}(r^{2n})$. Consequently, we choose an extension by setting

$$m_\pm(\mathfrak{z}, r, \omega) = r^2 m_2(\pm i + |r|\mathfrak{z}, |r|, \omega),$$

then $m_\pm$ is a $\mathscr{C}^1$ function of its arguments for $|\mathfrak{z}| \leq R_0/2$, $|r| < 1/2$, $\omega \in \mathbb{S}^{d-1}$, which moreover satisfies

$$m_\pm(\mathfrak{z}, 0, \omega) = \partial_r m_\pm(\mathfrak{z}, 0, \omega) = 0. \tag{4.25}$$

Let us set

$$F_\pm(\mathfrak{z}, r, \omega) = \pm 2i\mathfrak{z} + r\mathfrak{z}^2 - \frac{r}{(\pm i + r\mathfrak{z})^2}\left(3H_\mu\omega\cdot\omega + m_\pm(\mathfrak{z}, r, \omega)\right)$$

and observe that $F_\pm$ is a $\mathscr{C}^1$, $\mathbb{C}$ valued function of its arguments for $|\mathfrak{z}| \leq R_0/2$, $|r| < 1/2$, $\omega \in \mathbb{S}^{d-1}$ that coincides with $f_\pm$ if $r > 0$. Therefore it suffices to study the zeroes of $F_\pm$. For every $\omega \in \mathbb{S}^{d-1}$, using (4.25), $F_\pm(0, 0, \omega) = 0$ and

$$D_w F_\pm(0, 0, \omega) = \pm 2i$$

is invertible (as a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$). Therefore by the implicit function theorem, for every $\omega \in \mathbb{S}^{d-1}$, there exists a vicinity of $(0, 0, \omega)$ such that the zeroes of $F_\pm$ are given by a $\mathscr{C}^1$ curve. By compactness, we can then find ε_3 such that for every $|r| \leq \varepsilon_3$, $|\mathfrak{z}| \leq \varepsilon_3$, and $\omega \in \mathbb{S}^{d-1}$, the zeroes of $F_\pm(\cdot, r, \omega)$ are described by a curve $\mathfrak{z} = W_\pm(r, \omega)$ such that $W_\pm(0, \omega) = 0$. Since by using again (4.25), we have

$$\partial_r F_\pm(0, r, \omega) = 3H_\mu\omega\cdot\omega \neq 0,$$

and we also obtain that

$$\partial_r W_\pm(0, \omega) = \pm i\frac{3}{2}H_\mu\omega\cdot\omega.$$

This yields v). Note that

$$\partial_r \Gamma_\pm(0, \omega) = 0, \quad \partial_r \mathcal{T}_\pm(0, \omega) = \pm\frac{3}{2}H_\mu\omega\cdot\omega.$$

The fact that we necessarily have $\Gamma_\pm(r, \omega) < 0$ for $r > 0$ is a consequence of (H1). ☐

In the above proof, we have used the implicit function theorem in polar coordinates in order to describe the zeroes of $z^2 + 1 - m_{KE}$ in the region $|\gamma| \leq \varepsilon_3|\xi|$, $|\tau \pm 1| \leq \varepsilon_3|\xi|$ and $|\xi| \leq \varepsilon_3$. Nevertheless, it will be useful to get that $Z_\pm(r, \omega)$ are actually smooth functions of ξ under the additional symmetry assumption (2.2).

Lemma 4.2. *Assuming* (1.2), (1.4), (2.1), (2.2) *and (H1), there exists* $\varepsilon_3 > 0$ *such that for every* $\xi \neq 0$, *the zeroes of* $1 - m_{VP}$ *or equivalently of* $z^2 + 1 - m_{KE}$ *with* $|\gamma| \leq \varepsilon_3|\xi|$, $|\tau \pm 1| \leq \varepsilon_3|\xi|$ *and* $|\xi| \leq \varepsilon_3$ *are given by two smooth curves of class* $\mathscr{C}^{d+2}$ *under the form*

$$Z_\pm(\xi) = \pm i + i|\xi|^2\Phi_\pm(\xi)$$

where

$$\Phi_\pm(0) = \pm\frac{3}{2}C_\mu \in \mathbb{R},\ \mathrm{Im}\ \Phi_\pm \geq 0.$$

Proof. By using the notations of the proof of Proposition 4.2 v), since $W_\pm$ is $\mathscr{C}^1$ and $W_\pm(0, \omega) = 0$, we can set $W_\pm(r, \omega) = r\tilde{W}_\pm(r, \omega)$ and thanks to (4.24), we see that for $r \neq 0$, $\tilde{W}_\pm(r, \omega)$ is a zero of $\tilde{f}_\pm(\mathfrak{z}, r, \omega)$ where

$$\tilde{f}_\pm(\mathfrak{z}, r, \omega) = \pm 2i\mathfrak{z} + r^2\mathfrak{z}^2 - \frac{1}{(\pm i + r^2\mathfrak{z})^2}\left(3H_\mu\omega\cdot\omega + r^2 m_2(\pm i + r^2\mathfrak{z}, r, \omega)\right).$$

Moreover, thanks to (2.2), we have that

$$H_\mu\omega\cdot\omega = C_\mu$$

is independent of ω and that by using Lemma 4.1 and in particular the expansion (4.10) and (4.14), we infer that $r^4 m_2(\pm i + r^2\mathfrak{z}, r, \omega)$ has an expansion in terms of polynomials of r^2 of valuation larger than two plus a high order remainder of the form (4.12). We can therefore write

$$r^2 m_2(\pm i + r^2\mathfrak{z}, r, \omega) =: m_\pm(\mathfrak{z}, \xi)$$

where $m_\pm$ is a smooth function of its arguments such that $m_\pm(\mathfrak{z},0)=0$, $D_\mathfrak{z} m_\pm(\mathfrak{z},0)=0$. We can thus write $\tilde f_\pm$ as a smooth function of ξ:

$$\tilde f_\pm(\mathfrak{z},\xi)=\pm 2i\mathfrak{z}+|\xi|^2\mathfrak{z}^2-\frac{1}{(\pm i+|\xi|^2\mathfrak{z})^2}\left(3C_\mu+m_\pm(\mathfrak{z},\xi)\right).$$

Moreover, we observe that

$$\tilde f_\pm(\pm\frac{3}{2}iC_\mu,0)=0,\quad D_\mathfrak{z}\tilde f_\pm(\pm\frac{3}{2}iC_\mu,0)=\pm 2i.$$

Consequently, from the implicit function theorem we find that $\tilde W_\pm(r,\omega)$ is a smooth function of ξ that we still denote by $\tilde W_\pm(\xi)$. This yields

$$Z_\pm(r,\omega)=|\xi|^2\tilde W_\pm(\xi),\quad \tilde W_\pm(0)=\pm\frac{3}{2}iC_\mu,$$

which concludes the proof of the lemma. □

5. Kernel estimates

5.1. **Short time estimates.** We start with short time estimates, which require little assumption on μ.

Proposition 5.1. *Assuming* (1.4)*, there exists $C>0$ such that for every $t\in(0,1]$,*

$$\|G(t)\|_{L^1}\le Ct,\quad \|G(t)\|_{L^\infty}\le C\frac{1}{t^{d-1}}.$$

Proof. We observe that $G(t,x)$ solves the integral equation

$$G=K+K*_{t,x}G$$

$$\begin{aligned}K(t,x)=e^{\gamma t}\mathcal{F}^{-1}(m_{VP}(\gamma,\tau,\xi))(t,x)&=\int_{\mathbb{R}^d\times\mathbb{R}}e^{\gamma t+i\tau t+ix\cdot\xi}m_{VP}(\gamma,\tau,\xi)\,d\tau d\xi\\&=-\frac{1}{t^{d-1}}\mu\left(\frac{x}{t}\right)1_{t\ge 0}.\end{aligned}$$

Therefore, we have the estimate

$$\|K(t)\|_{L^1}\lesssim t.$$

Moreover, as already observed, G vanishes in the past, therefore

$$G(t,x)=K(t,x)+\int_0^t K(t-s,\cdot)*_x G(s,\cdot)\,ds,\quad \forall t\ge 0.$$

This yields

$$\|G(t)\|_{L^1}\lesssim t+\int_0^t(t-s)\|G(s)\|_{L^1}\,ds$$

and hence from the Gronwall inequality, we get that

$$\|G(t)\|_{L^1}\lesssim t,\quad \forall t\le 1.$$

We also obtain that

$$\|G(t)\|_{L^\infty}\lesssim\frac{1}{t^{d-1}}+\int_0^{\frac t2}\frac{1}{(t-s)^{d-1}}\|G(s)\|_{L^1}\,ds+\int_{\frac t2}^t(t-s)\|G(s)\|_{L^\infty}\,ds.$$

This yields for $t\le 1$ that $y(t)=t^{d-1}\|G(t)\|_{L^\infty}$ verifies

$$y(t)\lesssim 1+t^d\int_0^{\frac t2}\frac{1}{(t-s)^{d-1}}\,ds+t^{d-1}\sup_{[0,t]}y(s)\int_{\frac t2}^t(t-s)\frac{1}{s^{d-1}}\,ds.$$

Since

$$t^d \int_0^{\frac{t}{2}} \frac{1}{(t-s)^{d-1}} ds = t^2 \int_0^{\frac{1}{2}} \frac{1}{(1-u)^{d-1}}\, ds = Ct^2$$

and

$$t^{d-1} \int_{\frac{t}{2}}^t (t-s) \frac{1}{s^{d-1}}\, ds \leq 2^{d-1} t^2 \int_{\frac{1}{2}}^1 (1-u)\, du \lesssim t^2,$$

we get that for every $T > 0$

$$\sup_{[0,T]} y(t) \lesssim 1 + T^2 + T^2 \sup_{[0,T]} y(t).$$

This yields the result for $t \in (0, T]$, T sufficiently small. We can then iterate the argument finitely many times in a classical way to get the result for $t \in (0, 1]$. This ends the proof. $\square$

5.2. **Large time estimates.** We shall now focus on estimates for $t \geq 1$. First, observe that by setting $z = \gamma + i\tau$, we can write (3.3) as

$$G(t,x) = \frac{1}{i} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \left(\int_{\text{Re } z=\gamma} e^{zt} \frac{m_{VP}(z,\xi)}{1 - m_{VP}(z,\xi)}\, dz \right) d\xi.$$

Let us pick $\delta > 0$ to be fixed later. We split G as a high frequency and a low frequency part:

$$G(t,x) = G^H(t,x) + G^L(t,x) \tag{5.1}$$

where

$$G^H(t,x) = \frac{1}{i} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \left(\int_{\text{Re } z=\gamma} e^{zt} \frac{m_{VP}(z,\xi)}{1 - m_{VP}(z,\xi)} \left(1 - \chi\left(\frac{\xi}{\delta}\right)\right) dz \right) d\xi, \tag{5.2}$$

$$G^L(t,x) = \frac{1}{i} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \left(\int_{\text{Re } z=\gamma} e^{zt} \frac{m_{VP}(z,\xi)}{1 - m_{VP}(z,\xi)} \chi\left(\frac{\xi}{\delta}\right) dz \right) d\xi. \tag{5.3}$$

where $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ is a nonnegative radial function equal to one for $|\xi| \leq 1$ and supported in the ball of radius 2. Note that G^H and G^L depends on δ. The choice of δ will be carefully performed in order to estimate G^L.

5.2.1. *High frequency estimates.* We shall first estimate the high frequency contribution G^H.

Proposition 5.2. *Assuming* (1.2), (1.4), (2.1) *and (H1), for every $\delta > 0$, there exist $C > 0$ and $\alpha > 0$ such that*

$$\|G^H(t)\|_{L^1} \leq Ce^{-\alpha t}, \quad \|G^H(t)\|_{L^\infty} \leq Ce^{-\alpha t}, \quad \forall t \geq 1.$$

Proof. Let us first recall that $(1-\chi)$ is supported in the zone $|\xi| \geq \delta > 0$ so that the argument is very similar to the one used in the torus case in [12] for example. For $\xi \neq 0$, thanks to the Penrose stability condition (H1) and Corollary 4.1, the function $m_{VP}(\cdot,\xi)/(1 - m_{VP}(\cdot,\xi))$ is an holomorphic function in $\{\text{Re}\, z > 0\}$. Moreover, by using ii) of Proposition 4.2, for $|\xi| \geq \delta$, it extends as an holomorphic function in $\{\text{Re}\, z > -R_\delta|\xi|\}$, where R_δ is given by ii) of Proposition 4.2, and we have a positive uniform estimates from below of $|1 - m_{VP}|$. We can then use the Cauchy formula to get that for $|\xi| \geq \delta$,

$$\int_{\text{Re } z=\gamma} e^{zt} \frac{m_{VP}(z,\xi)}{1 - m_{VP}(z,\xi)} \chi\left(\frac{\xi}{\delta}\right) dz = \int_{\text{Re } z=-R_\delta|\xi|/2} e^{zt} \frac{m_{VP}(z,\xi)}{1 - m_{VP}(z,\xi)} \chi\left(\frac{\xi}{\delta}\right) dz.$$

Indeed, we can apply Corollary (4.1) to get that uniformly for $|\xi| \geq \delta$ and $\text{Re}\, z \geq -R_\delta|\xi|/2$,

$$|m_{VP}(z,\xi)| \lesssim \frac{C}{|\xi|^2 + \tau^2} \tag{5.4}$$

so that there is no contribution from infinity. Consequently, we have to estimate

$$G^H(t,x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} \int_{\mathbb{R}} e^{-R_\delta|\xi|t/2} e^{i\tau t} \frac{m_{VP}(-R_\delta|\xi|/2,\tau,\xi)}{1-m_{VP}(-R_\delta|\xi|/2,\tau,\xi)} \left(1-\chi\left(\frac{\xi}{\delta}\right)\right) d\tau d\xi.$$

By using again (5.4) and ii) of Proposition 4.2, we easily get that

$$|G^H(t,x)| \lesssim \int_{|\xi|\geq\delta} e^{-R_\delta|\xi|t/2} \int_{\mathbb{R}} \frac{1}{|\xi|^2+\tau^2}\, d\tau\, d\xi \lesssim \int_{|\xi|\geq\delta} e^{-R_\delta|\xi|t/2} \frac{1}{|\xi|}\, d\xi \lesssim e^{-\tilde{\alpha}t}$$

for some $\tilde{\alpha} > 0$. This yields

$$\|G^H(t)\|_{L^\infty} \lesssim e^{-\tilde{\alpha}t},\ \forall t\geq 1.$$

For the L^1 norm, by integrating by parts in ξ and applying Corollary 4.1, we obtain in a similar way that for all multi-indices $|\beta| \leq d+1$,

$$|x^\beta G^H(t,x)| \lesssim (1+t^{|\beta|})e^{-\tilde{\alpha}t}, \quad \forall t\geq 1.$$

Therefore, we obtain that

$$|G^H(t,x)| \leq \frac{1}{1+|x|^{d+1}}(1+t^{d+1})e^{-\tilde{\alpha}t}, \quad \forall t\geq 1$$

and hence that

$$\|G^H(t)\|_{L^1} \lesssim e^{-\alpha t}, \quad \forall t \geq 1,$$

with $\alpha = \tilde{\alpha}/2$. □

5.2.2. *Low frequency estimates.* We shall now estimate the low frequency part G^L.

Lemma 5.1. *Assuming* (1.2), (1.4), (2.1), *(H1), (H2), for $\delta > 0$ small enough, we have the following decomposition of G^L:*

$$G^L(t,x) = G^r(t,x) + G^S_+(t,x) + G^S_-(t,x) \tag{5.5}$$

where,

$$G^r(t,x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} \left(\int_{\mathrm{Re}\ z=-\tilde{\delta}|\xi|} e^{zt}\frac{m_{VP}(z,\xi)}{1-m_{VP}(z,\xi)}dz\right)\chi\left(\frac{\xi}{\delta}\right)\, d\xi, \quad \tilde{\delta} = \delta^{3/2},$$

$$G^S_\pm(t,x) = 2\pi\int_{\mathbb{R}^d} e^{ix\cdot\xi}e^{Z_\pm(r,\omega)t}a_\pm(\xi)\chi\left(\frac{\xi}{\delta}\right)\, d\xi, \quad a_\pm(\xi) = \frac{Z_\pm(r,\omega)^2}{2Z_\pm(r,\omega)-\partial_z m_{KE}(Z_\pm(r,\omega),\xi)}.$$

Proof. We now deal with the region $|\xi| \leq \delta$ for $\delta > 0$ to be chosen sufficiently small. For $\xi \neq 0$, we would like again to use the Cauchy formula to change the integration contour for

$$I_\xi = \frac{1}{i}\int_{\mathrm{Re}\ z=\gamma} e^{zt}\frac{m_{VP}(z,\xi)}{1-m_{VP}(z,\xi)}\chi\left(\frac{\xi}{\delta}\right)\, dz.$$

Again for $\gamma > 0$ the function $1-m_{VP}(z,\xi)$ does not vanish thanks to (H1) so that we have to carefully study what happens for negative γ with $|\gamma|$ small. We observe that thanks to i) and iii) of Proposition 4.2, for $|(\gamma,\tau,\xi)| \leq 2\varepsilon_1$ or $|(\gamma,\tau,\xi)| \geq 1/(2\varepsilon_1)$ (reducing ε_1 if necessary), the function $1-m_{VP}$ does not vanish in $\mathcal{C}_{R_1}$.

We shall now choose

$$0 < \delta \leq \frac{\varepsilon_3}{10A_0} \tag{5.6}$$

where ε_3 and A_0 are given by Proposition 4.2 iv) and v). As a consequence, for $|\xi| \leq \delta$, $|\gamma| \leq \delta|\xi|$, if $|\tau| \geq 1/\varepsilon_1$ or $|\tau| \leq \varepsilon_1$, $1-m_{VP}$ does not vanish. Then, since $\delta \leq \varepsilon_3$, we get that for $|\xi| \leq \delta$, $|\gamma| \leq \delta|\xi|$ and $|\tau \pm 1| \leq \varepsilon_3|\xi|$, the function $1-m_{VP}$ has for each ξ exactly two zeroes described by v) of Proposition 4.2. Moreover, since $\Gamma_\pm(0,\omega) = 0$, $\partial_r\Gamma_\pm(0,\omega) = 0$, we have that for $|\tau\pm 1| \leq \delta|\xi|$, the zeroes in $-\delta|\xi| \leq \gamma \leq 0$ are actually localized in $-\ C\delta|\xi|^2 \leq -C|\xi|^3 \leq \gamma \leq 0$ for some

$C > 0$. Therefore, assuming that δ is sufficiently small, we get in particular that on the line $\mathrm{Re}\, z = \gamma = -\delta^{\frac{3}{2}}|\xi|$, there is no zero of $1 - m_{VP}$ for $|\tau \pm 1| \leq \delta|\xi|$, $|\xi| \leq \delta$. Next, using iv) of Proposition 4.2 since $\delta \leq \varepsilon_2$, we get that for $|\gamma| \leq \delta^{\frac{3}{2}}|\xi|$, $|\tau \pm 1| \geq \delta|\xi|$, $\varepsilon_1 \leq |\tau| \leq 1/\varepsilon_1$ and $|\xi| \leq \delta$, $1 - m_{VP}$ does not vanish.

To summarize, we have thus obtained that for each $\xi \neq 0$, $|\xi| \leq \delta$, there are exactly two zeroes of $(1 - m_{VP})$ in the region $|\gamma| \leq \delta|\xi|$ and they are described by v) of Proposition 4.2. Moreover, they are localized in $-C\delta|\xi|^2 \leq \gamma \leq 0$ and $|\tau \pm 1| \leq \varepsilon_3|\xi|$. We can thus use the residue formula to write that for $\xi \neq 0$ (note that there is again no contribution from infinity since the estimate (5.4) is still valid for large τ),

$$I_\xi = \frac{1}{i}\int_{\mathrm{Re}\ z=-\delta^{\frac{3}{2}}|\xi|} e^{zt}\frac{m_{VP}(z,\xi)}{1-m_{VP}(z,\xi)}\chi\left(\frac{\xi}{\delta}\right)dz + 2\pi\chi\left(\frac{\xi}{\delta}\right)\sum_{\pm} e^{Z_\pm(r,\omega)t}\left(\mathrm{Res}\frac{m_{VP}}{1-m_{VP}}(\cdot,\xi)\right)_{|Z_\pm(r,\omega)}$$

where $r = |\xi|$, $\omega = \xi/|\xi|$. Computing the residue, we obtain

$$2\pi\sum_{\pm}\left(\mathrm{Res}\frac{m_{VP}}{1-m_{VP}}(\cdot,\xi)\right)_{|Z_\pm(r,\omega)} = -2\pi\sum_{\pm}\frac{1}{\partial_z m_{VP}(Z_\pm(\xi),\xi)}.$$

To get regularity in ξ close to $\xi = 0$, it is convenient to express the residue in terms of m_{KE}. Thanks to (4.4), we have

$$m_{VP} = -\frac{1}{z^2}(1 - m_{KE})$$

and hence

$$\partial_z m_{VP} = \frac{2}{z^3}(1 - m_{KE}) + \frac{1}{z^2}\partial_z m_{KE}.$$

Since $(1 - m_{KE}) = -z^2$ at $z = Z_\pm$, we can also write

$$2\pi\sum_{\pm}\left(\mathrm{Res}\frac{m_{VP}}{1-m_{VP}}(\cdot,\xi)\right)_{|Z_\pm(r,\omega)} = 2\pi\sum_{\pm}\frac{Z_\pm(r,\omega)^2}{2Z_\pm(r,\omega) - \partial_z m_{KE}(Z_\pm(r,\omega),\xi)}.$$

This yields the decomposition of G^L

$$G^L(t,x) = G^r(t,x) + G^S_+(t,x) + G^S_-(t,x) \tag{5.7}$$

where (setting $\tilde\delta = \delta^{\frac{3}{2}}$ for notational convenience),

$$G^r(t,x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi}\left(\int_{\mathrm{Re}\ z=-\tilde\delta|\xi|} e^{zt}\frac{m_{VP}(z,\xi)}{1-m_{VP}(z,\xi)}dz\right)\chi\left(\frac{\xi}{\delta}\right)d\xi,$$

$$G^S_\pm(t,x) = 2\pi\int_{\mathbb{R}^d} e^{ix\cdot\xi}e^{Z_\pm(r,\omega)t}a_\pm(\xi)\chi\left(\frac{\xi}{\delta}\right)d\xi, \quad a_\pm(\xi) = \frac{Z_\pm(r,\omega)^2}{2Z_\pm(r,\omega) - \partial_z m_{KE}(Z_\pm(r,\omega),\xi)},$$

giving the lemma. $\square$

5.2.3. *Low frequency estimates: regular part.* The next step is to estimate G^r and $G^S_\pm$ in (5.7). We start with G^r.

Proposition 5.3 (Study of G^r)**.** *Assuming* (1.2), (1.4), (2.1) *and (H2), δ can be chosen small enough so that uniformly for $t \geq 1$*

$$\|G^r\|_{L^\infty} \lesssim \frac{1}{t^{d+1}}, \quad \|G^r\|_{L^1} \lesssim \frac{1}{t}.$$

Proof. Let us write that

$$G^r(t,x) = \frac{1}{i}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\left(\int_{\mathrm{Re}\ z=-\tilde\delta|\xi|} e^{-\tilde\delta|\xi|t}e^{i\tau t}\frac{m_{VP}(z,\xi)}{1-m_{VP}(z,\xi)}dz\right)\chi\left(\frac{\xi}{\delta}\right)d\xi = \int_{\mathbb{R}^d} e^{ix\cdot\xi}\tilde I_\xi\chi\left(\frac{\xi}{\delta}\right)d\xi.$$

Thanks to Proposition 4.1, we observe that

$$\frac{m_{VP}(z,\xi)}{1-m_{VP}(z,\xi)} = -\frac{1}{z^2}\frac{1-m_{KE}(z,\xi)}{1+\frac{1}{z^2}(1-m_{KE}(z,\xi))} = -\frac{1-m_{KE}(z,\xi)}{z^2+1-m_{KE}(z,\xi)}$$
$$= -1 + \frac{z^2}{z^2+1-m_{KE}} = -1 + \frac{z^2}{z^2+1}\frac{1}{(1-\frac{m_{KE}}{1+z^2})} = -1 + \frac{z^2}{z^2+1} + \frac{z^2}{(z^2+1)^2}\frac{m_{KE}}{(1-\frac{m_{KE}}{1+z^2})}$$
$$= -\frac{1}{z^2+1} + \frac{z^2}{z^2+1}\frac{m_{KE}}{(z^2+1-m_{KE})}.$$

We can thus write

$$\tilde{I}_\xi = -\frac{1}{i}\int_{\mathrm{Re}\ z=-\tilde{\delta}|\xi|} e^{zt}\frac{1}{1+z^2}\,dz + J_\xi, \quad J_\xi = \frac{1}{i}\int_{\mathrm{Re}\ z=-\tilde{\delta}|\xi|} e^{zt}\frac{z^2}{z^2+1}\frac{m_{KE}}{(z^2+1-m_{KE})}(z,\xi)\,dz.$$

Next, we observe that from the Cauchy formula

$$\int_{\mathrm{Re}\ z=-\tilde{\delta}|\xi|} e^{zt}\frac{1}{1+z^2}\,dz = \int_{\mathrm{Re}\ z=-\Gamma} e^{zt}\frac{1}{1+z^2}dz$$

for any $\Gamma \geq \tilde{\delta}|\xi|$ and thus, sending Γ to $+\infty$, we get that

$$\int_{\mathrm{Re}\ z=-\tilde{\delta}|\xi|} e^{zt}\frac{1}{1+z^2}\,dz = 0.$$

We have thus obtained that

$$G^r(t,x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} J_\xi \chi\left(\frac{\xi}{\delta}\right)\,d\xi, \quad J_\xi = \frac{1}{i}\int_{\mathrm{Re}\ z=-\tilde{\delta}|\xi|} e^{zt}\frac{z^2}{z^2+1}\frac{m_{KE}}{(z^2+1-m_{KE})}(z,\xi)\,dz. \tag{5.8}$$

We shall use this more convenient form to prove the estimates. We shall further split the J_ξ term into

$$\begin{aligned} J_\xi = &\int_{|\tau|\leq\varepsilon_1} e^{-\tilde{\delta}|\xi|t}e^{i\tau t}\frac{z^2}{z^2+1}\frac{m_{KE}}{(z^2+1-m_{KE})}(z,\xi)\,d\tau \\ &+ \int_{|\tau|\geq\varepsilon_1,\,||\tau|-1|\geq 1/2} e^{-\tilde{\delta}|\xi|t}e^{i\tau t}\frac{z^2}{z^2+1}\frac{m_{KE}}{(z^2+1-m_{KE})}(z,\xi)\,d\tau \\ &+ \int_{|\tau|\geq\varepsilon_1,\,||\tau|-1|\leq\frac{1}{2}} e^{-\tilde{\delta}|\xi|t}e^{i\tau t}\frac{z^2}{z^2+1}\frac{m_{KE}}{(z^2+1-m_{KE})}(z,\xi)\,d\tau =: J_{\xi,1}+J_{\xi,2}+J_{\xi,3}, \end{aligned} \tag{5.9}$$

where we recall that ε_1 is defined in iii) of Proposition 4.2 and we decompose accordingly G^r into

$$G^r = G^r_1 + G^r_2 + G^r_3. \tag{5.10}$$

In the following three lemmas, we shall provide estimates of G^r_i, $i = 1, 2, 3$.

Let us start with G^r_2, which is the easiest one.

Lemma 5.2. *Under the assumptions of Proposition 5.3, δ can be chosen small enough so that uniformly for $t \geq 1$,*

$$\|G^r_2\|_{L^\infty} \lesssim \frac{1}{t^{d+1}}, \quad \|G^r_2\|_{L^1} \lesssim \frac{1}{t}.$$

Proof. By using the same factorization as in (4.22), we observe that uniformly for $|\xi| \leq \delta$, $||\tau|-1| \geq 1/2$ and $\gamma = -\tilde{\delta}|\xi|$, we can take δ small enough so that

$$|z^2+1-m_{KE}(-\tilde{\delta}|\xi|,\tau,\xi)| \geq \kappa_0|z^2+1| > 0, \tag{5.11}$$

where κ_0 depends only on δ and ε_1. Moreover, still in the same range of parameters,

$$|z^2+1| = |z+i||z-i| \gtrsim 1+\tau^2, \quad |z|^2 \lesssim |\xi|^2+\tau^2. \tag{5.12}$$

Therefore, we get that

$$|G_2^r(t,x)| \lesssim \int_{|\xi|\le\delta} e^{-\tilde{\delta}|\xi|t} \int_{\mathbb{R}} \frac{|\xi|^2+|\tau|^2}{1+|\tau|^4} |m_{KE}(-\tilde{\delta}|\xi|,\tau,\xi)|\,d\tau d\xi$$
$$\lesssim \int_{|\xi|\le\delta} e^{-\tilde{\delta}|\xi|t} \int_{\mathbb{R}} |m_{KE}(-\tilde{\delta}|\xi|,\tau,\xi)|\,d\tau d\xi.$$

We then set $\tau = |\xi|\tau'$ and use the degree zero homogeneity of m_{KE} to get

$$|G_2^r(t,x)| \lesssim \int_{|\xi|\le\delta} e^{-\tilde{\delta}|\xi|t}|\xi| \int_{\mathbb{R}} \left| m_{KE}\left(-\tilde{\delta},\tau',\frac{\xi}{|\xi|}\right)\right|\,d\tau' d\xi.$$

From Proposition 4.1, we have that $\left| m_{KE}\left(-\tilde{\delta},\tau',\frac{\xi}{|\xi|}\right)\right|$ is uniformly bounded for $|\tau'|\le 2$, while we get from Lemma 4.1

$$\left| m_{KE}\left(-\tilde{\delta},\tau',\frac{\xi}{|\xi|}\right)\right| \lesssim \frac{1}{(\tau')^2}, \quad |\tau'|\ge 2,$$

and hence

$$\left| m_{KE}\left(-\tilde{\delta},\tau',\frac{\xi}{|\xi|}\right)\right| \lesssim \frac{1}{1+(\tau')^2}. \tag{5.13}$$

This yields

$$|G_2^r(t,x)| \lesssim \int_{|\xi|\le\delta} e^{-\tilde{\delta}|\xi|t}|\xi| \int_{\mathbb{R}} \frac{1}{1+(\tau')^2}\,d\tau' d\xi.$$

and hence by finally setting $\tilde{\xi} = t\xi$, we obtain

$$|G_2^r(t,x)| \lesssim \frac{1}{t^{d+1}}.$$

There remains to estimate the L^1 norm. To this end, we use an homogeneous Littlewood-Paley decomposition. We write

$$G_2^r = \sum_{q\le 0} G_{2,q}^r, \quad G_{2,q}^r(t,x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} J_{\xi,2}(t)\chi\left(\frac{\xi}{\delta}\right)\phi\left(\frac{\xi}{2^q}\right)\,d\xi, \tag{5.14}$$
$$J_{\xi,2}(t) = \int_{|\tau|\ge\varepsilon_1,\,||\tau|-1|\ge 1/2} e^{-\tilde{\delta}|\xi|t} e^{i\tau t} \frac{z^2}{z^2+1} \frac{m_{KE}}{(z^2+1-m_{KE})}(-\tilde{\delta}|\xi|,\tau,\xi)\,d\tau, \quad z = -\tilde{\delta}|\xi| + i\tau,$$

where ϕ is supported in the annulus $1/4\le|\xi|\le 4$. Changing ξ for $\xi/2^q$, we get that

$$G_{2,q}^r(t,x) = 2^{qd}\mathcal{G}_{2,q}^r(T,X), \quad \mathcal{G}_{2,q}^r(T,X) = \int_{\mathbb{R}^d} e^{iX\cdot\xi} J_{\xi,2,q}(T)\chi\left(\frac{2^q\xi}{\delta}\right)\phi(\xi)\,d\xi, \quad T = 2^q t,\ X = 2^q x, \tag{5.15}$$
$$J_{\xi,2,q}(T) = \int_{|\tau|\ge\varepsilon_1,\,||\tau|-1|\ge 1/2} e^{-\tilde{\delta}|\xi|T} e^{\frac{i\tau T}{2^q}} \frac{z^2}{z^2+1} \frac{m_{KE}}{(z^2+1-m_{KE})}(-\tilde{\delta}2^q|\xi|,\tau,2^q\xi)\,d\tau,$$

where now the integral in ξ is supported on the annulus $1/4\le|\xi|\le 4$. We then observe that since $2^q\le 1$, we have that

$$\|\mathcal{G}_q(T,\cdot)\|_{L^1} \lesssim \sum_{|\alpha|\le d+1} \|\partial_\xi^\alpha J_{\xi,2,q}\|_{L^\infty}.$$

By using again (5.11), (5.12) and Proposition 4.1, we get that for $|\alpha|\le d+1$, and uniformly for $1/4\le|\xi|\le 4$,

$$|\partial_\xi^\alpha J_{\xi,2,q}(T)| \lesssim \int_{||\tau|-1|\ge 1/2} e^{-\tilde{\delta}|\xi|T} \sum_{k\le|\alpha|} 2^{qk}|h_k(-\tilde{\delta}2^q|\xi|,\tau,2^q\xi)|\,d\tau,$$

where h_k is positively homogeneous of degree $-k$. Moreover, by using the expansion of m_{KE} provided by Lemma 4.1, we also know that for bounded ξ and $|\tau| \geq 1/2$,

$$|h_k(z,\xi)| \lesssim \frac{1}{|z^2+1|} \lesssim \frac{1}{1+\tau^2}.$$

Therefore, by setting $\tau = 2^q|\xi|\tau'$, we obtain that

$$|\partial_\xi^\alpha J_{\xi,2,q}(T)| \lesssim 2^q \int_{|2^q|\xi|\tau'|-1|\geq 1/2} e^{-\tilde{\delta}|\xi|T} \sum_{k\leq|\alpha|} |h_k(-\tilde{\delta},\tau',\frac{\xi}{|\xi|})|\, d\tau'$$
$$\lesssim 2^q e^{-\tilde{\delta}T/2} \int_{\mathbb{R}} \frac{1}{1+(\tau')^2}\, d\tau' \lesssim 2^q e^{-\tilde{\delta}T/2}.$$

Consequently we obtain from (5.15) that

$$\|G_{2,q}^r(t)\|_{L^1} \lesssim 2^q e^{-\tilde{\delta}2^q t/2} \lesssim \frac{2^q}{1+(2^q t)^4}.$$

This finally yields for $t \geq 1$

$$\|G_2^r(t)\|_{L^1} \leq \sum_{q\leq 0} \|G_{2,q}^r(t)\|_{L^1} \lesssim \sum_{2^q\leq 1/t} 2^q + \frac{1}{t^4} \sum_{1/t\leq 2^q\leq 0} \frac{1}{2^{3q}} \lesssim \frac{1}{t} + \frac{t^3}{t^4} \lesssim \frac{1}{t}.$$

This ends the proof. □

Let us turn to G_1^r.

Lemma 5.3. *Under the assumptions of Proposition 5.3, we have uniformly for $t \geq 1$*

$$\|G_1^r\|_{L^\infty} \lesssim \frac{1}{t^{d+2}}, \quad \|G_1^r\|_{L^1} \lesssim \frac{1}{t^2}.$$

Proof. In this regime of low frequencies for $|\xi| \leq \delta$, $|\tau| \leq \varepsilon_1$, and $\gamma = -\tilde{\delta}|\xi|$, we have that

$$\left| \frac{z^2}{z^2+1} \frac{m_{KE}}{z^2+1-m_{KE}}(-\tilde{\delta}|\xi|,\tau,\xi) \right| \lesssim |z|^2 \left| \frac{m_{KE}(-\tilde{\delta}|\xi|,\tau,\xi)}{z^2+1-m_{KE}(-\tilde{\delta}|\xi|,\tau,\xi)} \right|.$$

By setting again $\tau = |\xi|\tau'$, and by using that m_{KE} is homogeneous of degree zero, we obtain that

$$|G_1^r(t,x)| \lesssim \int_{|\xi|\leq\delta} e^{-\tilde{\delta}|\xi|t}|\xi|^3 \int_{|\tau'|\leq\varepsilon_1/|\xi|} |z'|^2 \left| \frac{m_{KE}(-\tilde{\delta},\tau',\frac{\xi}{|\xi|})}{|\xi|^2(z')^2+1-m_{KE}(-\tilde{\delta},\tau',\frac{\xi}{|\xi|})} \right| d\tau'$$

where $z' = -\tilde{\delta}+i\tau'$. By using Proposition 4.2 iii), in particular (4.17), we know that

$$\left| |\xi|^2(z')^2+1-m_{KE}(-\tilde{\delta},\tau',\frac{\xi}{|\xi|}) \right|$$

is bounded from below by a positive constant since $|z'| \geq \tilde{\delta}$ and hence, obtain that

$$|G_1^r(t,x)| \lesssim \int_{|\xi|\leq\delta} e^{-\tilde{\delta}|\xi|t}|\xi|^3 \int_{|\tau'|\leq\varepsilon_1/|\xi|} |z'|^2 \left| m_{KE}(-\tilde{\delta},\tau',\frac{\xi}{|\xi|}) \right| d\tau'.$$

As in (5.13), we have

$$\left| m_{KE}\left(-\tilde{\delta},\tau',\frac{\xi}{|\xi|}\right) \right| \lesssim \frac{1}{1+(\tau')^2}.$$

This yields in particular that $|z'|^2 \left|m_{KE}(-\tilde\delta,\tau',\frac{\xi}{|\xi|})\right|$ is uniformly bounded for $\tau'\in\mathbb{R}$. Therefore, we obtain that

$$|G_1^r(t,x)| \lesssim \int_{|\xi|\le\delta} e^{-\tilde\delta|\xi|t}|\xi|^3 \int_{|\tau'|\le\varepsilon_1/|\xi|} d\tau'\,d\xi \lesssim \frac{1}{t^{d+2}}.$$

To estimate the L^1 norm, we argue as in the proof of Lemma 5.2, writing

$$\text{(5.16)}\quad G_1^r=\sum_{q\le 0} G_{1,q}^r, \qquad G_{1,q}^r(t,x)=2^{qd}\mathcal{G}_{1,q}^r(T,X),$$

$$\mathcal{G}_{1,q}^r(T,X)=\int_{\mathbb{R}^d} e^{iX\cdot\xi}J_{\xi,1,q}(T)\chi\left(\frac{2^q\xi}{\delta}\right)\phi\left(\xi\right)\,d\xi, \quad T=2^q t,\ X=2^q x,$$

$$J_{\xi,1,q}(T)=\int_{|\tau|\le\varepsilon_1} e^{-\tilde\delta|\xi|T}e^{\frac{i\tau T}{2^q}}\frac{z^2}{z^2+1}\frac{m_{KE}}{(z^2+1-m_{KE})}(-\tilde\delta 2^q|\xi|,\tau,2^q\xi)\,d\tau,$$

where the integral in ξ is supported on the annulus $1/4\le|\xi|\le 4$. We use again that since $2^q\le 1$, we have the estimate

$$\|\mathcal{G}_{1,q}(T,\cdot)\|_{L^1}\lesssim \sum_{|\alpha|\le d+1}\|\partial_\xi^\alpha J_{\xi,1,q}\|_{L^\infty}.$$

From the same estimates as above, we obtain that for $1/4\le|\xi|\le 4$,

$$|\partial_\xi^\alpha J_{\xi,1,q}(T)|\lesssim \int_{|\tau|\le\varepsilon_1}|z|^2 e^{-\tilde\delta|\xi|T}\sum_{k\le|\alpha|}2^{qk}|h_k(-\tilde\delta 2^q|\xi|,\tau,2^q\xi)|\,d\tau$$

where h_k is positively homogeneous of degree $-k$. Arguing exactly as in the proof of Lemma 5.2, we get

$$|\partial_\xi^\alpha J_{\xi,2,q}(T)|\lesssim 2^{3q}\int_{\tau'|\le\varepsilon_1/(2^q|\xi|)} e^{-\tilde\delta|\xi|T}\sum_{k\le|\alpha|}(1+(\tau')^2)|h_k(-\tilde\delta,\tau',\frac{\xi}{|\xi|})|\,d\tau'\lesssim 2^{2q}e^{-\tilde\delta T/2}.$$

We can then conclude as in the proof of the previous lemma by summing over the dyadic blocks. The proof is complete. □

It remains to estimate G_3^r.

Lemma 5.4. *Under the assumptions of Proposition 5.3, δ can be chosen small enough so that uniformly for $t\ge 1$*

$$\|G_3^r\|_{L^\infty}\lesssim\frac{1}{t^{d+1}},\quad \|G_3^r\|_{L^1}\lesssim\frac{1}{t}.$$

Proof. We are now integrating on $||\tau|-1|\le 1/2$. We shall decompose $J_{\xi,3}$ from (5.9) as

$$J_{\xi,3}=J_{\xi,3,0}+\sum_{1\le k\le N}J_{\xi,3,k}$$

where

$$J_{\xi,3,0}=\int_{||\tau|-1|\le\varepsilon_3|\xi|} e^{-\tilde\delta|\xi|t}e^{i\tau t}\frac{z^2}{z^2+1}\frac{m_{KE}}{(z^2+1-m_{KE})}(z,\xi)\,d\tau$$

and for $1\le k\le N$,

$$J_{\xi,3,k}=\int_{2^k\varepsilon_3|\xi|\le||\tau|-1|\le 2^{k+1}\varepsilon_3|\xi|} e^{-\tilde\delta|\xi|t}e^{i\tau t}\frac{z^2}{z^2+1}\frac{m_{KE}}{(z^2+1-m_{KE})}(z,\xi)\,d\tau,$$

where ε_3 is given by v) of Proposition 4.2, and N is such that $2^{N+1}\varepsilon_3|\xi|\le\frac12$.

Let us first estimate $J_{\xi,3,0}$. We shall focus on the estimates close to $\tau = 1$ (and call the corresponding term $J^+_{\xi,3,0}$), the ones close to $\tau = -1$ can be obtained from the same arguments. We set

$$J^+_{\xi,3,0} = \int_{|\tau-1|\leq\varepsilon_3|\xi|} e^{-\tilde{\delta}|\xi|t} e^{i\tau t} \frac{z^2}{z^2+1} \frac{m_{KE}}{(z^2+1-m_{KE})}(z,\xi)\, d\tau, \quad z = -\tilde{\delta}|\xi| + i\tau.$$

We first set $\tau = 1 + |\xi|\tau'$ so that

$$J^+_{\xi,3,0} = |\xi| \int_{|\tau'|\leq\varepsilon_3} e^{|\xi|\mathfrak{z}t} e^{it} \frac{(i+|\xi|\mathfrak{z})^2}{2i|\xi|\mathfrak{z} + |\xi|^2\mathfrak{z}^2} \frac{m_{KE}(i+|\xi|\mathfrak{z},\xi)}{|\xi| f_+(\mathfrak{z},|\xi|,\omega)}\, d\tau',$$

where we have set $\mathfrak{z} = (z-i)/|\xi| = -\tilde{\delta} + i\tau'$ and $\omega = \xi/|\xi|$ so that $\tau' = \operatorname{Im}\mathfrak{z}$ and f_+ is defined in (4.24). As in (4.23), we can write

$$m_{KE}(i+|\xi|\mathfrak{z},\xi) = \frac{r^2}{(i+r\mathfrak{z})^2}(3H_\mu\omega\cdot\omega + r^2 m_2(i+r\mathfrak{z},r,\omega)), \quad r = |\xi|,$$

therefore, we have

$$|m_{KE}(i+|\xi|\mathfrak{z},\xi)| \lesssim |\xi|^2.$$

Moreover, since

$$f_+(\mathfrak{z},r,\omega) = 2i\mathfrak{z} + r\mathfrak{z}^2 - \frac{r}{(i+r\mathfrak{z})^2}(3H_\mu\omega\cdot\omega + r^2 m_2(i+r\mathfrak{z},r,\omega)),$$

we observe that for $\operatorname{Re}\mathfrak{z} = -\tilde{\delta} = -\delta^{\frac{3}{2}}$ and $r \leq \delta$ (see again the proof of Lemma 5.1), we have for δ small enough that

$$|f_+(\mathfrak{z},r,\omega)| \gtrsim 1.$$

By using also that for δ sufficiently small,

$$\left|2i|\xi|\mathfrak{z} + |\xi|^2\mathfrak{z}^2\right| \gtrsim |\xi|,$$

this yields,

$$|J^+_{\xi,3,0}| \lesssim |\xi| e^{-\tilde{\delta}|\xi|t}.$$

The same arguments apply for $J^-_{\xi,3,0}$. We thus have

$$|J_{\xi,3,0}| \lesssim |\xi| e^{-\tilde{\delta}|\xi|t}. \tag{5.17}$$

Let us now estimate $J_{\xi,3,k}$. Thanks to iv) of Proposition 4.2, we have

$$|z^2+1-m_{KE}| \gtrsim 2^k\varepsilon_3|\xi|. \tag{5.18}$$

As above, the estimate

$$|m_{KE}| \lesssim |\xi|^2,$$

still holds, and since $\frac{1}{2} \geq ||\tau|-1| \geq 2^k\varepsilon_3|\xi|$, we also have

$$\frac{1}{|z^2+1|} \lesssim \frac{1}{2^k\varepsilon_3|\xi|},$$

therefore, we obtain that

$$|J_{\xi,3,k}| \lesssim e^{-\tilde{\delta}|\xi|t} \frac{2^k}{2^{2k}}|\xi| \lesssim e^{-\tilde{\delta}|\xi|t}|\xi|\frac{1}{2^k}. \tag{5.19}$$

By combining, (5.17) and (5.19), we thus obtain

$$|J_{\xi,3}(t,x)| \lesssim e^{-\tilde{\delta}|\xi|t}|\xi| \sum_{k\geq 0} \frac{1}{2^k} \lesssim e^{-\tilde{\delta}|\xi|t}|\xi|.$$

From the definition of G_3^r (see (5.10), (5.9)), we finally obtain

$$|G_3^r(t,x)| \lesssim \frac{1}{t^{d+1}}.$$

To estimate the L^1 norm, we argue again as in the proof of Lemma 5.2, writing

$$G_3^r = \sum_{q\le 0} G_{3,q}^r, \quad G_{3,q}^r(t,x) = 2^{qd}\mathcal{G}_{3,q}^r(T,X), \tag{5.20}$$

$$\mathcal{G}_{3,q}^r(T,X) = \int_{\mathbb{R}^d} e^{iX\cdot\xi} J_{\xi,3}(T)\chi\left(\frac{2^q\xi}{\delta}\right)\phi(\xi)\,d\xi, \quad T = 2^q t,\ X = 2^q x,$$

$$J_{\xi,3,q}(T) = \int_{||\tau|-1|\le 1/2} e^{-\tilde\delta|\xi|T} e^{\frac{i\tau T}{2^q}} \frac{z^2}{z^2+1}\frac{m_{KE}}{(z^2+1-m_{KE})}(-\tilde\delta 2^q|\xi|,\tau,2^q\xi)\,d\tau,$$

where the integral in ξ is in the annulus $1/4 \le |\xi| \le 4$. To estimate $J_{\xi,3,q}(T)$, we focus again on the vicinity of 1 and call the corresponding contribution $J_{\xi,3,q}^+(T)$. We now use the same decomposition as before for the estimate of the L^∞ norm, which yields

$$\begin{aligned} J_{\xi,3,q}^+(T) &= \sum_{0\le k\le N} J_{\xi,3,q,k}^+(T) \\ &= \sum_{0\le k\le N}\int_{2^{k+q}|\xi|\varepsilon_3 \le |\tau-1| \le \varepsilon_3|\xi|2^{k+1+q}} e^{-\tilde\delta|\xi|T} e^{\frac{i\tau T}{2^q}} \frac{z^2}{z^2+1}\frac{m_{KE}}{(z^2+1-m_{KE})}\left(-\tilde\delta 2^q|\xi|,\tau,2^q\xi\right)\,d\tau. \end{aligned}$$

Let us estimate $\|\partial_\xi^\alpha J_{\xi,3,q}^+(T)\|_{L^\infty}$ for $|\alpha| \le d+1$.

By (4.23), we have that for $1/4 \le |\xi| \le 4$, $2^{k+q}|\xi|\varepsilon_3 \le |\tau-1| \le \varepsilon_3|\xi|2^{k+1+q}$ or $|\tau-1| \le \varepsilon_3 2^{2+q}$ for $k=0$,

$$m_{KE}(-\tilde\delta 2^q|\xi|,\tau,2^q\xi) = \frac{2^{2q}}{(-\tilde\delta 2^q|\xi| + i\tau)^2}\left(3H_\mu\xi\cdot\xi + |\xi|^4 m_2(-\tilde\delta 2^q|\xi|,\tau,2^q\xi)\right)$$

and hence for $|\tau-1| \le 1/2$ and $|\xi| \le \delta$, we get

$$\left|\partial_\xi^\beta\left(\frac{m_{KE}}{g}(-\tilde\delta 2^q|\xi|,\tau,2^q\xi)\right)\right| \lesssim \sum_{\ell\le|\beta|+1}\frac{2^{2q+q(\ell-1)}}{\left|g\left(-\tilde\delta 2^q|\xi|,\tau,2^q\xi\right)\right|^\ell},$$

where we have set

$$g(-\tilde\delta 2^q|\xi|,\tau,2^q\xi) = (z^2+1-m_{KE})(-\tilde\delta 2^q|\xi|,\tau,2^q\xi).$$

Consequently, by using (5.18) which gives for $|\xi| \ge 1/4$

$$|g(-\tilde\delta 2^q|\xi|,\tau,2^q\xi)| \gtrsim 2^{k+q},$$

we obtain that

$$\left|\partial_\xi^\beta\left(\frac{m_{KE}}{g}(-\tilde\delta 2^q|\xi|,\tau,2^q\xi)\right)\right| \lesssim \frac{2^q}{2^k}.$$

In a similar way, we have uniformly in q,

$$\left|\partial_\xi^\beta\left(\frac{(-\tilde\delta 2^q|\xi|+i\tau)^2}{(-\tilde\delta 2^q|\xi|+i\tau)^2+1}\right)\right| \lesssim \frac{1}{2^{k+q}}.$$

Therefore, we obtain that for $|\alpha| \le d+1$.

$$\|\partial_\xi^\alpha J_{\xi,3,q}^+(T)\|_{L^\infty} \lesssim \frac{2^{k+q}}{2^{2k}}e^{-\tilde\delta T/2} \le \frac{2^q}{2^k}e^{-\tilde\delta T/2}.$$

The same estimate holds for $J_{\xi,3,q}^-(T)$. By summing over $k \ge 0$, we get that

$$\|\mathcal{G}_{3,q}^r(T)\|_{L^1} \lesssim 2^q e^{-\tilde\delta T/2}$$

and we finally obtain the claimed estimate of $\|G_3^r(t)\|_{L^1}$ by summing over $q \leq 0$ as in the proof of Lemma 5.2.

$\square$

End of the proof of Proposition 5.3. It suffices to recall the expression (5.10) and to gather the estimates of Lemma 5.3, Lemma 5.2 and Lemma 5.4 (taking δ small enough). $\square$

5.2.4. *Low frequency estimates: singular part.* We shall now study $G_\pm^S$ defined in (5.7), which corresponds to the dispersive part.

Proposition 5.4. *Assuming* (1.2), (1.4), (2.1) *and* (2.2), *δ can be chosen small enough so that*

$$\|G_\pm^S(t)\|_{L^2} \leq C, \quad \|G_\pm^S(t)\|_{L^\infty} \leq \frac{C}{t^{\frac{d}{2}}}, \quad \forall t \geq 1 \tag{5.21}$$

where C depends on at most $d+1$ derivatives of the amplitude $a_\pm$ and $d+2$ derivatives of the phase $Z_\pm$. We also have the more precise structure

$$G_\pm^S(t) *_x \cdot = e^{\pm it} H_\pm^S(t, D) \tag{5.22}$$

where the operator $H_\pm^S(t, D)$ is such that for every $k \geq 0$,

$$\partial_t^k H_\pm^S(t, D) = H_{\pm,k}^S(t, D)\chi(D)\Delta^k, \tag{5.23}$$

and $H_{\pm,k}^S(t, D)$ also satisfy the estimates

$$\|H_{\pm,k}^S(t, D)\|_{L^2 \to L^2} \leq C, \quad \|H_{\pm,k}^S(t, D)\|_{L^1 \to L^\infty} \leq \frac{C}{t^{\frac{d}{2}}}, \quad \forall t \geq 1. \tag{5.24}$$

Proof. We focus on the study of G_+^S, the analysis of G_-^S being similar. The estimate for the L^2 norm is just a consequence of the fact that the inverse Fourier transform is an isometry. We recall

$$G_+^S(t, x) = \int_{\mathbb{R}^d} e^{Z_+(r,\omega)t + ix\cdot\xi} a_+(r, \omega)\chi\left(\frac{\xi}{\delta}\right) d\xi.$$

Since we assume that (2.2) holds, we can use Lemma 4.2, from which we deduce that Z_+ is a smooth function of the ξ variable in $B(0, 10\delta)$ so that we actually have

$$G_+^S(t, x) = \int_{\mathbb{R}^d} e^{Z_+(\xi)t + ix\cdot\xi} a_+(\xi)\chi\left(\frac{\xi}{\delta}\right) d\xi$$

where the the amplitude

$$a_+(\xi) = \frac{Z_+(\xi)^2}{2Z_+(\xi) - \partial_z m_{KE}(Z_+(\xi), \xi)}$$

is also a smooth function of ξ.

To get the decay estimate in L^∞, we shall use that the imaginary part of Z_+ described in Lemma 4.2 provides dispersive properties. Since we have almost no information on the real part of Z_+ (besides the fact that it is non-negative), we shall use a robust version of the stationary phase. By using Lemma 4.2, we can write

$$G_+^S(t, x) =: e^{it} H_+^S(t, x) = e^{it} I(t, X), \quad X = x/t$$

where

$$H_+^S(t, x) = I(t, X) = \int_{\mathbb{R}^d} e^{it\Psi_X(\xi)} a_+(\xi)\chi\left(\frac{\xi}{\delta}\right) d\xi \tag{5.25}$$

with the phase given by

$$\Psi_X(\xi) = |\xi|^2\Phi_+(\xi) + X\cdot\xi = \Psi_X^r(\xi) + i\Psi_X^i(\xi).$$

Note that $\Psi_X^i \geq 0$ and

$$D_\xi^2 \Psi_X^r(0) = 2C_\mu \mathrm{I}_d.$$

We can take δ small enough so that

$$D_\xi^2 \Psi_X^r(\xi) \geq \frac{1}{2} C_\mu \mathrm{I}_d \geq c_0 > 0 \tag{5.26}$$

for $|\xi| \leq 10\delta$ and hence that for every ξ_1, $\xi_2 \in B(0, 10\delta)$,

$$|\nabla \Psi_X^r(\xi_1) - \nabla \Psi_X^r(\xi_2)| \geq \tilde{c_0} |\xi_1 - \xi_2|, \tag{5.27}$$

where the lower bound is independent of X. We will rely on the approach of Lemma 3.1 of [8] by checking that the imaginary part is harmless. We use the operator

$$\mathrm{L}(u) = \frac{1}{i(1 + t|\nabla \Psi_X|^2)} \sum_{j=1}^d \partial_j \overline{\Psi_X} \partial_j u + \frac{1}{(1 + t|\nabla \Psi_X|^2)} u$$

(where $|\cdot|$ denotes in this context the hermitian norm of $\mathbb{C}^d$), which satisfies by construction

$$\mathrm{L}(e^{it\Psi_X}) = e^{it\Psi_X} \tag{5.28}$$

and has a formal adjoint $\widetilde{\mathrm{L}}$ (i.e. $\int_{\mathbb{R}^d} \mathrm{L}uv = \int_{\mathbb{R}^d} u \widetilde{\mathrm{L}} v$, $\forall u, v \in \mathscr{C}_c^\infty$) given by

$$\widetilde{\mathrm{L}}(u) = -\sum_{j=1}^d \frac{\partial_j \overline{\Psi_X}}{i(1 + t|\nabla \Psi_X|^2)} \partial_j u + \Big(-\sum_{j=1}^d \frac{\partial_j^2 \overline{\Psi_X}}{i(1 + t|\nabla \Psi_X|^2)} + \sum_{j=1}^d \frac{2t \partial_j \overline{\Psi_X} \, \mathrm{Re}(\nabla \Psi_X \cdot \nabla \partial_j \overline{\Psi_X})}{i(1 + t|\nabla \Psi_X|^2)^2} \Big) u + \frac{1}{(1 + t|\nabla \Psi_X|^2)} u.$$

Using (5.28) repeatedly, we thus get that

$$|I(t, X)| \lesssim \int_{\mathbb{R}^d} \left| (\widetilde{\mathrm{L}})^N \left(a_+(\cdot) \chi\left(\frac{\cdot}{\delta}\right)\right)\right| d\xi,$$

for any integer $N \geq 1$. We can then check that we get as in the proof of Lemma 3.1 in [8] that

$$(\widetilde{\mathrm{L}})^N = \sum_{|\alpha| \leq N} a_\alpha^{(N)} \partial^\alpha$$

where the coefficients $a_\alpha^{(N)}$ satisfy on the support of the amplitude the estimate

$$|a_\alpha^{(N)}| \leq C(\Lambda_{N+1}) \frac{1}{\langle t^{\frac{1}{2}} \nabla \Psi_X \rangle^N}$$

with

$$\Lambda_k = \sup_{\xi \in B(0, 5\delta)} \sup_{2 \leq |\alpha| \leq k} |\partial^\alpha \Psi_X|.$$

Note that since Λ_k involves only derivatives of order larger than 2 of Ψ_X, this quantity is independent of X. Then, by choosing $N = d + 1$, we get

$$|I(t, X)| \lesssim C(\Lambda_{N+1}, A_N) \int_{B(0,\delta)} \frac{1}{\langle t^{\frac{1}{2}} \nabla \Psi_X \rangle^N} d\xi$$

with $A_N = \sup_{|\alpha| \leq N} \|\partial^\alpha a_+\|_{L^\infty(B(0,\delta))}$. To conclude, we just use that

$$\int_{B(0,\delta)} \frac{1}{\langle t^{\frac{1}{2}} \nabla \Psi_X \rangle^N} d\xi \leq \int_{B(0,\delta)} \frac{1}{\langle t^{\frac{1}{2}} \nabla \Psi_X^r \rangle^N} d\xi.$$

We finally observe that by (5.26), (5.27), the map $\xi \mapsto \nabla\Psi^r_X$ is a diffeomorphism on $B(0,\delta)$ and we can thus use the change of variables $\eta = \nabla\Psi^r_X$ and apply the bound from below of the Jacobian provided by (5.26) to get

$$|I(t,X)| \lesssim C(\Lambda_{N+1}, A_N) \int_{\mathbb{R}^d} \frac{1}{(1+t|\eta|^2)^{N/2}}\, d\eta \lesssim C(\Lambda_{N+1}, A_N) \frac{1}{t^{\frac{d}{2}}}.$$

This yields (5.21).

To get (5.23), it suffices to notice that each time we take a time derivative of $H^S_\pm$ (see (5.25)), we multiply the amplitude by $i|\xi|^2\Psi_+(\xi)$. Since Ψ_+ is smooth the new amplitude has the same properties as before. The expression (5.22) follows by switching from kernels to operators. The proof is finally complete. □

6. Proof of Theorem 3.1

We use (5.1), take δ small enough so that to apply Lemma 5.1 and the estimates of Proposition 5.3 and Proposition 5.4, and finally apply Proposition 5.2. Theorem 3.1 follows, with the "regular" part of the kernel given by

$$G^R = G^H + G^r.$$

7. Proof of Theorem 2.1

From the method of characteristics, if $f(t,x,v)$ solves (1.3), then $\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\, dv$ solves (3.1) and hence (3.2) with

$$S(t,x) = \int_{\mathbb{R}^d} f_0(x - vt, v)\, dv.$$

We have the well-known dispersive estimates (see e.g. [2])

$$\begin{aligned} &\|S(t)\|_{L^1} \lesssim \|f_0\|_{L^1_{x,v}}, \quad \|S(t)\|_{L^\infty} \lesssim \frac{1}{t^d}\|f_0\|_{L^1_x L^\infty_v}, \quad t \geq 1, \\ &\|\nabla S(t)\|_{L^1} \lesssim \frac{1}{t}\|\nabla_x f_0\|_{L^1_{x,v}}, \quad \|\nabla S(t)\|_{L^\infty} \lesssim \frac{1}{t^{d+1}}\|\nabla_x f_0\|_{L^1_x L^\infty_v}, \quad t \geq 1. \end{aligned} \tag{7.1}$$

We then decompose

$$\rho(t,x) = \rho^R(t,x) + \rho^S_+(t,x) + \rho^S_-(t,x)$$

where $\rho^R(t,x)$ (resp. $\rho^S_\pm$) solves

$$\rho^R = S + G^R *_{t,x} S, \qquad \rho^S_\pm = G^S_\pm *_{t,x} S.$$

Note from Theorem 3.1 that G^R verifies the same estimates as the kernel of the linearized screened Vlasov-Poisson system (see Theorem 2.1 in [13]). Therefore, we obtain the same result as in Corollary 2.1 in [13]:

$$\|\rho^R(t)\|_{L^1} + t^d\|\rho^R(t)\|_{L^\infty} \lesssim \log(1+t)\left(\|f_0\|_{L^1_{x,v}} + \|f_0\|_{L^1_x L^\infty_v}\right), \quad \forall t \geq 1,$$

and we shall thus focus on the singular part $\rho^S_\pm$. We analyse the + case, the other one being similar.

The basic estimate consists in writing, thanks to (5.21) in Proposition 5.4,

$$\|\rho^S_+(t)\|_{L^\infty} \lesssim \int_0^{\frac{t}{2}} \frac{1}{(t-s)^{\frac{d}{2}}}\|S(s)\|_{L^1}\, ds + \int_{\frac{t}{2}}^t \|S(s)\|_{L^2}\, ds \lesssim \frac{1}{t^{\frac{d}{2}-1}}, \ \forall t \geq 1,$$

which does not decay for $d = 1, 2$. Assuming additionally that $\langle v\rangle\nabla_x f_0 \in L^1_{x,v}$, $\langle v\rangle\nabla_x f_0 \in L^1_x L^\infty_v$, we can improve this estimate by using the refined formula (5.22). We have

$$G^S_+ *_{t,x} S = \int_0^t e^{i(t-s)} H^S_+(t-s, D) S(s)\, ds.$$

By using that $i\partial_s e^{i(t-s)} = e^{i(t-s)}$, we can integrate by parts in time to get

$$G_+^S *_{t,x} S = i(H_+^S(0,D)S(t) - H_+^S(t,D)S(0)) \\ + i\int_0^t e^{i(t-s)}\partial_t H_+^S(t-s,D)S(s)\,ds - i\int_0^t e^{i(t-s)}H_+^S(t-s,D)\partial_s S(s)\,ds.$$

To estimate

$$\Sigma_1(t) = i(H_+^S(0,D)S(t) - H_+^S(t,D)S(0)),$$

we can use (5.24) and (7.1). This yields

$$\|\Sigma_1(t)\|_{L^\infty} \lesssim \frac{1}{t^{\frac{d}{2}}}\|f_0\|_{L^1_{x,v}}.$$

For

$$\Sigma_2(t) = i\int_0^t e^{i(t-s)}\partial_t H_+^S(t-s,D)S(s)\,ds - i\int_0^t e^{i(t-s)}H_+^S(t-s,D)\partial_s S(s)\,ds,$$

we can rely on (5.23) (since $\chi(D)$ is a Fourier multiplier with compactly supported symbol, we shall use that $\chi(D)\Delta$ can be bounded by $\chi(D)|\nabla|$). This entails

$$\|\Sigma_2(t)\|_{L^\infty} \lesssim \int_0^{\frac{t}{2}} \frac{1}{(t-s)^{\frac{d}{2}}}(\|\nabla S(s)\|_{L^1} + \|\partial_t S\|_{L^1})\,ds + \int_{\frac{t}{2}}^t (\|\nabla S(s)\|_{L^2} + \|\partial_t S(s)\|_{L^2})\,ds.$$

We therefore need to study decay estimates for $\partial_t S$ in L^1 and L^2. To this end, observe that

$$\partial_t S(t,x) = -\nabla\cdot J(t,x)$$

with

$$J(t,x) = \int_{\mathbb{R}^d} v f^l(t,x,v)\,dv$$

where $f^l(t,x,v)$ solves the free transport equation

$$\partial_t f^l + v\cdot\nabla_x f^l = 0$$

with initial data f_0. As for (7.1), we get that

$$\|\nabla J(t)\|_{L^\infty} \lesssim \frac{1}{t}\|\langle v\rangle\nabla_v f_0\|_{L^1_{x,v}}, \quad \|\nabla J(t)\|_{L^\infty} \lesssim \frac{1}{t^{d+1}}\|\langle v\rangle\nabla_v f_0\|_{L^1_x L^\infty_v}.$$

We thus get by interpolation that

$$\|\partial_t S(t)\|_{L^\infty} \lesssim \frac{1}{t^{d+1}}\|\langle v\rangle\nabla_v f_0\|_{L^1_{x,v}}, \quad \|\partial_t S(t)\|_{L^1} \lesssim \frac{1}{t}\|\langle v\rangle\nabla_v f_0\|_{L^1_x},$$
$$\|\partial_t S(t)\|_{L^2} \lesssim \frac{1}{t^{\frac{d}{2}+1}}\left(\|\langle v\rangle\nabla_v f_0\|_{L^1_{x,v}} + \|\langle v\rangle\nabla_v f_0\|_{L^1_x L^\infty_v}\right).$$

Using also (7.1), we deduce

$$\|\Sigma_2(t)\|_{L^\infty} \lesssim \left(\frac{1}{t^{\frac{d}{2}}}\int_0^{t/2}\frac{1}{\langle s\rangle}\,ds + \int_{\frac{t}{2}}^t \frac{1}{s^{\frac{d}{2}+1}}\,ds\right)\left(\|\langle v\rangle\nabla_v f_0\|_{L^1_{x,v}} + \|\langle v\rangle\nabla_v f_0\|_{L^1_x L^\infty_v}\right)$$
$$\lesssim \frac{\log(1+t)}{t^{\frac{d}{2}}}\left(\|\langle v\rangle\nabla_v f_0\|_{L^1_{x,v}} + \|\langle v\rangle\nabla_v f_0\|_{L^1_x L^\infty_v}\right).$$

This finally yields

$$\|\rho_+^S(t)\|_{L^\infty} \lesssim \frac{\log(1+t)}{t^{\frac{d}{2}}}\left(\|\langle v\rangle\nabla_v f_0\|_{L^1_{x,v}} + \|\langle v\rangle\nabla_v f_0\|_{L^1_x L^\infty_v}\right)$$

and the proof of Theorem 2.1 is complete.

8. Appendix: Radial decreasing equilibria satisfy the stability assumption (H2)

In this section we shall prove

Proposition 8.1. *Let μ satisfy* (1.2) *and* (1.4). *If $\mu(v) = F\left(\frac{|v|^2}{2}\right)$ with $F'(s) < 0$, $\forall s \geq 0$, then (H2) is verified.*

Proof. We study the function $\tau \mapsto 1 - m_{KE}(i\tau, \eta)$ for $\eta \in \mathbb{S}^{d-1}$. By using (4.10), we get that $m_{KE}(i\tau, \eta) \to 0$ when $|\tau|$ tends to $+\infty$, so that it suffices to study $\tau \mapsto 1 - m_{KE}(i\tau, \eta)$ for bounded τ. We have for $\gamma > 0$, $|\eta| = 1$,

$$m_{KE}(z, \eta) = -\int_0^{+\infty} e^{-(\gamma+i\tau)s}\, i\eta \cdot \sum_{k,l} \eta_k \eta_l \mathcal{F}_v(v_k v_l \nabla_v \mu)(\eta s)\, ds,$$

$$= -i \int_0^{+\infty} \int_{\mathbb{R}^d} e^{-(\gamma+i\tau+i\eta\cdot v)t} (\eta \cdot v)^3 F'\left(\frac{|v|^2}{2}\right) dvdt.$$

We then write $v = u\eta + w$ with $w \in \eta^\perp = H_\eta$ so that

$$m_{KE}(z, \eta) = -i \int_0^{+\infty} \int_{\mathbb{R}} e^{-(\gamma+i\tau+iu)t} u^3 \Phi'\left(\frac{u^2}{2}\right) dudt$$

where

$$\Phi(s) = \int_{H_\eta} F\left(s + \frac{|w|^2}{2}\right) dw. \tag{8.1}$$

This yields

$$m_{KE}(z, \eta) = -\int_{\mathbb{R}} \frac{\tau + u}{\gamma^2 + (\tau+u)^2} u^3 \Phi'\left(\frac{u^2}{2}\right) du - i\gamma \int_{\mathbb{R}} \frac{u^3}{\gamma^2 + (\tau+u)^2} \Phi'\left(\frac{u^2}{2}\right) du.$$

Taking the limit $\gamma \to 0$ (following e.g. [15, Proof of Prop. 2.1]), we get that

$$m_{KE}(i\tau, \eta) = -\text{p.v.} \int_{\mathbb{R}} \frac{u^3 \Phi'\left(\frac{u^2}{2}\right)}{\tau + u} du - i\pi\tau^3 \Phi'\left(\frac{\tau^2}{2}\right).$$

We then observe that for bounded τ the imaginary part vanishes only for $\tau = 0$ and in this case the real part is equal to

$$-\int_{\mathbb{R}} u^2 \Phi'\left(\frac{u^2 + |w|^2}{2}\right) du = \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} F\left(\frac{u^2 + |w|^2}{2}\right) dwdu = \int_{\mathbb{R}^d} \mu dv = 1.$$

Therefore $1 - m_{KE}(i\tau, \eta)$ vanishes only for $\tau = 0$.

Let us now compute $\partial_z m_{KE}(0, \eta)$ and $\partial_z^2 m_{KE}(0, \eta)$. Following the same lines, we first get that

$$\partial_z m_{KE}(z, \eta) = i \int_0^{+\infty} \int_{\mathbb{R}} t e^{-(\gamma+i\tau+iu)t} u^3 \Phi'\left(\frac{u^2}{2}\right) dudt$$

$$= \int_0^{+\infty} \int_{\mathbb{R}} e^{-(\gamma+i\tau+iu)t} \partial_u \left(u^3 \Phi'\left(\frac{u^2}{2}\right)\right) dudt$$

and therefore

$$\partial_z m_{KE}(z,\eta) = -i\int_{\mathbb{R}} \frac{\tau+u}{\gamma^2+(\tau+u)^2}\partial_u\left(u^3\Phi'\left(\frac{u^2}{2}\right)\right)du$$
$$+\gamma\int_{\mathbb{R}} \frac{1}{\gamma^2+(\tau+u)^2}\partial_u\left(u^3\Phi'\left(\frac{u^2}{2}\right)\right)du.$$

Taking the limit $\gamma \to 0$ as before, we get

$$\partial_z m_{KE}(0,\eta) = -i\int_{\mathbb{R}} \frac{1}{u}\partial_u\left(u^3\Phi'\left(\frac{u^2}{2}\right)\right)du.$$

By integrating by parts as before, this yields

$$\partial_z m_{KE}(0,\eta) = i\int_{\mathbb{R}} u\Phi'\left(\frac{u^2}{2}\right)du = 0.$$

Finally, for $\partial_z^2 m_{KE}(0,\eta)$, we have

$$\partial_z^2 m_{KE}(z,\eta) = -\int_0^{+\infty}\int_{\mathbb{R}} e^{-(\gamma+i\tau+iu)t}t\partial_u\left(u^3\Phi'\left(\frac{u^2}{2}\right)\right)dudt$$
$$= i\int_0^{+\infty}\int_{\mathbb{R}} e^{-(\gamma+i\tau+iu)t}t\partial_u^2\left(u^3\Phi'\left(\frac{u^2}{2}\right)\right)dudt.$$

This yields as before

$$\partial_z^2 m_{KE}(0,\eta) = -\int_{\mathbb{R}} \frac{1}{u}\partial_u^2\left(u^3\Phi'\left(\frac{u^2}{2}\right)\right)du = \int_{\mathbb{R}} \frac{1}{u^2}\partial_u\left(u^3\Phi'\left(\frac{u^2}{2}\right)\right)du = 2\int_{\mathbb{R}} \Phi'\left(\frac{u^2}{2}\right)du.$$

By using the definition (8.1), we thus get that

$$\partial_z^2 m_{KE}(0,\eta) = 2\int_{\mathbb{R}^d} F'\left(\frac{|v|^2}{2}\right)dv \neq 0$$

and the proof of the proposition is complete. ☐

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