QUASINEUTRAL LIMIT FOR VLASOV-POISSON WITH PENROSE STABLE DATA

by

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Abstract. — We study the quasineutral limit of a Vlasov-Poisson system that describes the dynamics of ions in a plasma. We handle data with Sobolev regularity under the sharp assumption that the profiles in velocity of the initial data satisfy a Penrose stability condition.

As a byproduct of our analysis, we obtain a well-posedness theory for the limit equation (which is a Vlasov equation with Dirac measure as interaction kernel), for such data.

Résumé. — Nous étudions la limite quasineutre d'un système de Vlasov-Poisson qui décrit la dynamique d'ions dans un plasma. Nous travaillons avec des données à régularité Sobolev sous l'hypothèse optimale que les profils en vitesse des données initiales satisfont une condition de stabilité de Penrose.

Comme corollaire de notre analyse, nous obtenons une théorie d'existence et d'unicité pour l'équation limite (qui est une équation de Vlasov avec une mesure de Dirac pour noyau d'interaction), pour de telles données.

Contents

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1. Introduction and main results

We study the quasineutral limit, that is the limit $\varepsilon \to 0$, for the following Vlasov-Poisson system describing the dynamics of ions in the presence of massless electrons:

$$\begin{cases} \partial_t f_\varepsilon \,+\, v\cdot\nabla_x f_\varepsilon \,+\, E_\varepsilon\cdot\nabla_v f_\varepsilon \,=\, 0, \\ E_\varepsilon = -\nabla_x V_\varepsilon, \\ V_\varepsilon - \varepsilon^2 \Delta V_\varepsilon = \displaystyle\int_{\mathbb{R}^d} f_\varepsilon \, dv - 1, \\ f_\varepsilon|_{t=0} = f_\varepsilon^0. \end{cases} \tag{1.1}$$

In these equations, the function $f_\varepsilon(t,x,v)$ stands for the distribution functions of the ions in the phase space $\mathbb{T}^d \times \mathbb{R}^d$, $d \in \mathbb{N}^*$, with $\mathbb{T}^d := \mathbb{R}^d/(2\pi\mathbb{Z})^d$. We assumed that the density of the electrons n_e satisfies a linearized Maxwell-Boltzmann law, that is $n_e = e^{V_\varepsilon} \sim 1 + V_\varepsilon$, which accounts for the source $-(1+V_\varepsilon)$ in the Poisson equation. Such a model was recently studied for instance in [**19, 20, 21, 11**]. Though we have focused on this simplified law, the arguments in this paper could be easily adapted to the model where the potential is given by the Poisson equation $-\varepsilon^2\Delta V_\varepsilon = \int_{\mathbb{R}^d} f_\varepsilon \, dv - e^{V_\varepsilon}$.

The dimensionless parameter ε is defined by the ratio between the Debye length of the plasma and the typical observation length. It turns out that in most practical situations, ε is very small, so that the limit $\varepsilon \to 0$, which bears the name of quasineutral limit, is relevant from the physical point of view. Observe that in the regime of small ε, we formally have that the density of ions is almost equal to that of electrons, hence the name quasineutral. This regime is so fundamental that it is even sometimes included in the very definition of a plasma, see e.g. [**8**].

The quasineutral limit for the Vlasov-Poisson system with the Poisson equation

$$-\varepsilon^2 \Delta V_\varepsilon = \int_{\mathbb{R}^d} f_\varepsilon \, dv - \int_{\mathbb{T}^d\times\mathbb{R}^d} f_\varepsilon \, dv \, dx, \tag{1.2}$$

that describes the dynamics of electrons in a fixed neutralizing background of ions is also very interesting. Nevertheless, we shall focus in this paper on the study of (1.1). The study of (1.2) combines the difficulties already present in this paper linked to kinetic instabilities and those related to high frequency waves due to the large electric field that, do not occur in the case of (1.1). The study of the combination of these two phenomena is postponed to future work.

It is straightforward to obtain the formal quasineutral limit of (1.1) as $\varepsilon \to 0$: we expect that $\varepsilon^2 \Delta V_\varepsilon$ tends to zero and hence if f_ε converges in a reasonable way to some f, then f should solve

$$\begin{cases} \partial_t f \,+\, v\cdot\nabla_x f \,+\, E\cdot\nabla_v f \,=\, 0, \\ E = -\nabla_x \rho, \quad \rho = \displaystyle\int_{\mathbb{R}^d} f \, dv, \\ f|_{t=0} = f^0. \end{cases} \tag{1.3}$$

This system was named *Vlasov-Dirac-Benney* by Bardos [**1**] and studied in [**3, 2**]. It was also referred to as the *kinetic Shallow Water system* in [**20**] by analogy with the classical Shallow Water system of fluid mechanics. In particular, it was shown in [**3**] that the semigroup of the linearized system around *unstable* equilibria is unbounded in Sobolev spaces (even with loss of derivatives). This yields the ill-posedness of (1.3) in Sobolev spaces, see in particular the recent work [**24**]. In [**2**], it was nevertheless shown in dimension one, i.e. for $d = 1$ that (1.3) is well-posed in the class of functions $f(x, v)$ such that for all $x \in \mathbb{T}$, $v \mapsto f(x, v)$ is compactly supported and is increasing for $v \leq m(t, x)$ and then decreasing for $v \geq m(t, x)$, that is to say for functions that for all x have the shape of *one bump*. The method in [**2**] is to reduce the problem to an infinite number of fluid type equations by using a water bag decomposition.

The mathematical study of the quasineutral limit started in the nineties with pioneering works of Brenier and Grenier for Vlasov with the Poisson equation (1.2), first with a limit involving defect measures [**7, 14**], then with a full justification of the quasineutral limit for initial data with uniform analytic regularity [**15**]. The work [**15**] also included a description of the so-called plasma waves, which are time oscillations of the electric field of frequency and amplitude $O(\frac{1}{\varepsilon})$. As already said, such oscillations actually do not occur in the quasineutral limit of (1.1). More recently, in [**22, 23**], relying on Wasserstein stability estimates inspired from [**25, 29**], it was proved that exponentially small but rough perturbations are allowed in the main result of [**15**].

In analytic regularity, it turns out that instabilities for the Vlasov-Poisson system, such as two-stream instabilities, do not have any effect, whereas in the class of Sobolev functions, they definitely play a crucial role. It follows that the quasineutral approximation both for (1.1) and (1.2) is not always valid. In particular, the convergence of (1.1) to (1.3) does not hold in general: we refer to [**16, 21**].

Nevertheless, it can be expected that the formal limit can be justified in Sobolev spaces for stable situations. We shall soon be more explicit about what we mean by stable data, but this should at least be included in the class of data for which the expected limit system (1.3) is well-posed. The first result in this direction is due to Brenier [**6**] (see also [**31**] and [**20**]), in which he justifies the quasineutral limit for initial data converging to a monokinetic distribution, that is a Dirac mass in velocity. This corresponds to a stable though singular case since the Dirac mass can be seen as an extremal case of a Maxwellian, that is a function with one bump. Brenier introduced the so-called modulated energy method to prove this result. Note that in this case the limit system is a fluid system (the incompressible Euler equations in the case of (1.2) or the shallow water equations in the case of (1.1)) and not a kinetic equation. This result is coherent with the fact that the instabilities present at the kinetic level do not show up at the one-fluid level, for example the quasineutral limit of the Euler-Poisson system can be justified in Sobolev spaces as shown for example in [**9, 28**], among others.

For non singular stable data with Sobolev regularity, there are only few available results which all concern the one-dimensional case $d = 1$.

- In [**21**], using the modulated energy method, the quasineutral limit is justified for very special initial data namely initial data converging to one bump functions that are furthermore *symmetric* and space homogeneous (thus that are stationary solutions to (1.1) and (1.3)). It is also proved that this is the best we could hope for with this method.
- Grenier sketched in [**16**] a result of convergence for data such that for every x the profile in v has only one bump. The proposed proof involves a functional taking advantage of the monotonicity in the one bump structure. Such kind of functionals have been recently used in other settings, for example in the study of the hydrostatic Euler equation or the Prandtl equation, see for example [**32, 33, 12**].

The main goal of this work is to justify the quasineutral limit that is to prove the derivation of (1.3) from (1.1) in the general stable case and in any dimension. As we shall see below a byproduct of the main result is the well-posedness of the system (1.3) in any dimension for smooth data with finite Sobolev regularity such that for every x, the profile $v \mapsto f^0(x, v)$ satisfies a Penrose stability condition. This condition is automatically satisfied in dimension one by smooth functions that for every x have a "one bump" profile, as well as by small perturbations of such functions.

To state our results, we shall first introduce the Penrose stability condition [**37**] for homogeneous equilibria $\mathbf{f}(v)$. Let us define for the profile $\mathbf{f}$ the Penrose function

$$\mathcal{P}(\gamma, \tau, \eta, \mathbf{f}) = 1 - (2\pi)^d \int_0^{+\infty} e^{-(\gamma + i\tau)s} \, \frac{i\eta}{1 + |\eta|^2} \cdot (\mathcal{F}_v \nabla_v \mathbf{f})(\eta s) \, ds,$$

for $\gamma > 0$, $\tau \in \mathbb{R}$, $\eta \in \mathbb{R}^d \backslash \{0\}$. The normalization factor $(2\pi)^d$ comes from our convention for the Fourier transform below. We shall say that the profile $\mathbf{f}$ satisfies the Penrose stability condition if

$$\inf_{(\gamma,\tau,\eta) \in (0,+\infty) \times \mathbb{R} \times \mathbb{R}^d \backslash \{0\}} |\mathcal{P}(\gamma, \tau, \eta, \mathbf{f})| > 0. \tag{1.4}$$

It will be also convenient to say that $\mathbf{f}$ satisfies the c_0 Penrose stability condition for some $c_0 > 0$ if

$$\inf_{(\gamma,\tau,\eta) \in (0,+\infty) \times \mathbb{R} \times \mathbb{R}^d \backslash \{0\}} |\mathcal{P}(\gamma, \tau, \eta, \mathbf{f})| \geq c_0. \tag{1.5}$$

The non-vanishing of $\mathcal{P}$ only has to be checked on a compact subset if f is smooth and localized enough (for example if $f \in \mathcal{H}^3_\sigma$, $\sigma > d/2$ with the notation below) and thus if $\mathbf{f}$ verifies the stability condition (1.4) then it also satisfies the stability condition (1.5) for some $c_0 > 0$. This condition is necessary for the large time stability of the profile $\mathbf{f}$ in the unscaled Vlasov-Poisson equation (that is to say for (1.1) with $\varepsilon = 1$). Note that it was recently proven in [**36, 5**] that Landau damping holds in small Gevrey neighborhood of such stable solutions, which means that these profiles are nonlinearly stable and even asymptotically stable in a suitable sense with respect to small perturbations in Gevrey spaces (see also [**10**] for Landau damping in Sobolev spaces for the Vlasov-HMF equations).

Remark 1. — *The assumption* (1.5) *is automatically satisfied in a small data regime. In a one-dimensional setting,* (1.5) *is also satisfied for the one bump profiles described previously. More generally, in any dimension,* (1.5) *is verified for any radial non-increasing function (therefore, Maxwellians are included) and there exist more sophisticated criteria based on the one bump structure of the averages of the function along hyperplanes. We refer to* [**36**] *for other conditions ensuring* (1.5). *Note that any sufficiently small perturbation of a Penrose stable profile is also Penrose stable.*

Throughout this paper, we consider the following normalization for the Fourier transform on $\mathbb{T}^d$ and $\mathbb{R}^m$ (m will usually be d or $d+1$):

$$\mathcal{F}(u)(k) := (2\pi)^{-d} \int_{\mathbb{T}^d} u(x) e^{-ik\cdot x}\, dx, \quad k \in \mathbb{Z}^d,$$
$$\mathcal{F}(w)(\xi) := (2\pi)^{-m} \int_{\mathbb{R}^m} w(y) e^{-i\xi\cdot y}\, dy, \quad \xi \in \mathbb{R}^m.$$

We will also often use the notation $\hat{\cdot}$ for $\mathcal{F}(\cdot)$. With this convention, the inverse Fourier transform yields

$$u(x) = \sum_{k\in\mathbb{Z}^d} \mathcal{F}(u)(k) e^{ik\cdot x}, \quad x \in \mathbb{T}^d,$$
$$w(y) = \int_{\mathbb{R}^m} \mathcal{F}(w)(\xi) e^{i\xi\cdot y}\, d\xi, \quad y \in \mathbb{R}^m.$$

For $k \in \mathbb{N}, r \in \mathbb{R}$, we introduce the weighted Sobolev norms

$$\|f\|_{\mathcal{H}^k_r} := \left(\sum_{|\alpha|+|\beta|\le k} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} (1+|v|^2)^r |\partial_x^\alpha \partial_v^\beta f|^2\, dv dx \right)^{1/2}, \tag{1.6}$$

where for $\alpha = (\alpha_1, \cdots, \alpha_d), \beta = (\beta_1, \cdots, \beta_d) \in \mathbb{N}^d$, we write

$$|\alpha| = \sum_{i=1}^d \alpha_i, \quad |\beta| = \sum_{i=1}^d \beta_i,$$
$$\partial_x^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}, \quad \partial_v^\beta := \partial_{v_1}^{\beta_1} \cdots \partial_{v_d}^{\beta_d}.$$

We will also use the standard Sobolev norms, for functions $\rho(x)$ that depend only on x

$$\|\rho\|_{H^k} = \|\rho\|_{H^k_x} := \left(\sum_{|\alpha|\le k} \int_{\mathbb{T}^d} |\partial_x^\alpha \rho|^2\, dx \right)^{1/2}. \tag{1.7}$$

Setting $\rho_\varepsilon(t,x) := \int_{\mathbb{R}^d} f_\varepsilon(t,x,v)\,dv$, we introduce the key quantity

$$\mathcal{N}_{2m,\,2r}(t,f_\varepsilon) := \|f_\varepsilon\|_{L^\infty((0,t),\mathcal{H}^{2m-1}_{2r})} + \|\rho_\varepsilon\|_{L^2((0,t),H^{2m})}, \tag{1.8}$$

where $m \in \mathbb{N}^*$ and $r \in \mathbb{R}_+$ are parameters that will be taken sufficiently large.

Let us finally fix our regularity indices. We define

$$m_0 = 3 + \frac{d}{2} + p_0, \quad p_0 = \lfloor \frac{d}{2} \rfloor + 1, \quad r_0 = \max(d, 2 + \frac{d}{2}) \tag{1.9}$$

and we shall mainly work with $2m > m_0$ and $2r > r_0$.

The main result of this paper is a uniform in ε local existence result in Sobolev spaces for (1.1) in the case of data for which the profile $v \mapsto f^0(x,v)$ satisfy the Penrose stability condition (1.4) for every x. More precisely, we shall prove the following theorem.

Theorem 1. — *Assume that for all $\varepsilon \in (0,1]$, $f^0_\varepsilon \in \mathcal{H}^{2m}_{2r}$ with $2m > m_0$, $2r > r_0$ and that there is $M_0 > 0$ such that for all $\varepsilon \in (0,1]$, $\|f^0_\varepsilon\|_{\mathcal{H}^{2m}_{2r}} \leq M_0$. Assume moreover that there is $c_0 > 0$ such that for every $x \in \mathbb{T}^d$ and for every $\varepsilon \in (0,1]$, the profile $v \mapsto f^0_\varepsilon(x,v)$ satisfies the c_0 Penrose stability condition* (1.5).

Then there exist $T > 0$, $R > 0$ (independent of ε) and a unique solution $f_\varepsilon \in \mathcal{C}([0,T],\mathcal{H}^{2m}_{2r})$ of (1.1) *such that*

$$\sup_{\varepsilon\in(0,1]} \mathcal{N}_{2m,2r}(T,f_\varepsilon) \leq R$$

and $f_\varepsilon(t,x,\cdot)$ satisfies the $c_0/2$ Penrose stability condition (1.5) *for every $t \in [0,T]$ and $x \in \mathbb{T}^d$.*

As already mentioned, the Penrose stability condition that we assume is sharp in the sense that it is necessary in order to justify the quasineutral limit for data with Sobolev regularity, see [**21**]. By Remark 1, the assumption that for all $x \in \mathbb{T}^d$, $v \mapsto f^0_\varepsilon(x,v)$ satisfies the c_0 Penrose stability condition (1.5) is verified if f^0_ε converges to a function f^0 under the form

$$f^0(x,v) = F(x,|v-u(x)|^2) + g(x,v)$$

where $F : \mathbb{T}^d \times \mathbb{R}_+ \to \mathbb{R}_+$ is such that for every x, $F(x,\cdot)$ is non-increasing, u is a smooth function and g is a sufficiently small perturbation (in $\mathcal{H}^s_\sigma$ for $s > 2$ and $\sigma > d/2$). This class includes in particular small perturbations of smooth local Maxwellians:

$$M(x,v) = \frac{\rho(x)}{(2\pi T(x))^{d/2}} \exp\left(-\frac{|v-u(x)|^2}{T(x)}\right).$$

Note also that though it is not stated in the Theorem, the solution f_ε remains in $\mathcal{H}^{2m}_{2r}$ on $[0,T]$. Nevertheless, the $\mathcal{H}^{2m}_{2r}$ norm is not controlled uniformly in ε, only the quantity $\mathcal{N}_{2m,\,2r}(T,f_\varepsilon)$ is.

From this uniform existence result, we are then able to justify the quasineutral limit for (1.1).

Theorem 2. — *Let $f_\varepsilon^0 \in \mathcal{H}^{2m}_{2r}$ satisfying the assumptions of Theorem 1. Assume that in addition there is $f^0 \in L^2(\mathbb{T}^d \times \mathbb{R}^d)$ such that $f_\varepsilon^0 \to f^0$ in $L^2(\mathbb{T}^d \times \mathbb{R}^d)$. Then, on the interval $[0,T]$ with $T > 0$ defined in Theorem 1, we have that*

$$\sup_{[0,T]} \|f_\varepsilon - f\|_{L^2_{x,v} \cap L^\infty_{x,v}} + \sup_{[0,T]} \|\rho_\varepsilon - \rho\|_{L^2_x \cap L^\infty_x} \to_{\varepsilon \to 0} 0$$

where f is a solution of (1.3) with initial data f^0 such that $f \in \mathcal{C}([0,T], \mathcal{H}^{2m-1}_{2r})$, $\rho = \int_{\mathbb{R}^d} f dv \in L^2([0,T], H^{2m})$ and that satisfies the $c_0/2$ Penrose stability condition (1.5) for every $t \in [0,T]$ and $x \in \mathbb{T}^d$.

As a byproduct of our analysis, we obtain local well-posedness for Vlasov-Dirac-Benney, in the class of Penrose stable initial data. Existence is a consequence of the statement of Theorem 2, while uniqueness is more subtle and is rather a consequence of the analysis that is used to prove Theorem 1.

Theorem 3. — *Let $f^0 \in \mathcal{H}^{2m}_{2r}$ with $2m > m_0$, $2r > r_0$ be such that for all $x \in \mathbb{T}^d$, $v \mapsto f^0(x,v)$ satisfies the c_0 Penrose stability condition (1.5). Then, there exists $T > 0$ for which there is a unique solution to (1.3) with initial condition f^0 and such that $f \in \mathcal{C}([0,T], \mathcal{H}^{2m-1}_{2r})$, $\rho \in L^2([0,T], H^{2m})$ and $v \mapsto f(t,x,v)$ satisfies the $c_0/2$ Penrose condition for every $t \in [0,T]$ and $x \in \mathbb{T}^d$.*

Remark 2. — *We have focused on periodic boundary conditions in x. Nevertheless, our results could be extended to the case $x \in \mathbb{R}^d$ without major changes.*

2. Strategy

Let us explain in this section the general strategy that we follow in this paper. The main part of this work consists of Sections 3, 4, 5 and 6 where we provide the proof of Theorem 1.

Our proof is based on a bootstrap argument, which we initiate in Section 3. The main difficulty is to derive a suitable uniform estimate for

$$\mathcal{N}_{2m,\,2r}(T, f_\varepsilon) = \|f_\varepsilon\|_{L^\infty((0,T), \mathcal{H}^{2m-1}_{2r})} + \|\rho_\varepsilon\|_{L^2((0,T), H^{2m})},$$

for some $T > 0$. Note that if we consider data with a better localization in the velocity space, we could rely on the fact that for all $\varepsilon > 0$, there is a unique global classical solution of (1.1) (see [**4**]). However, such a result is not useful in view of the quasineutral limit, since it does not provide estimates that are uniform in ε.

Assuming a control of $\|\rho_\varepsilon\|_{L^2((0,t),H^{2m})}$ for some $t > 0$, the estimate for $\|f_\varepsilon\|_{L^\infty((0,t),\mathcal{H}^{2m-1}_{2r})}$ can be obtained from a standard energy estimate (see Lemma 4). Consequently, the difficulty is to estimate $\|\rho_\varepsilon\|_{L^2((0,t),H^{2m})}$. From now on, we will forget the subscript ε to reduce the amount of notation.

A natural idea would be to use the fact that up to commutators, given $f(t,x,v)$ satisfying (1.1), $\partial_x^{2m} f$ evolves according to the linearized equation about f, that is

$$\partial_t \partial_x^{2m} f + v \cdot \nabla_x \partial_x^{2m} f + \partial_x^{2m} E \cdot \nabla_v f + E \cdot \nabla_v \partial_x^{2m} f + \cdots = 0, \tag{2.1}$$

where $\cdots$ should involve remainder terms only. One thus has first to understand this linearized equation. When $f \equiv \overline{f}(v)$ does not depend on t and x, then the linearized equation reduces to

$$\partial_t \dot{f} + v \cdot \nabla_x \dot{f} + \dot{E} \cdot \nabla_v \overline{f} = S, \quad \dot{E} = -\nabla_x (I - \varepsilon^2 \Delta_x)^{-1} \int_{\mathbb{R}^d} \dot{f} \, dv, \tag{2.2}$$

and one can deduce an integral equation for $\dot{\rho} = \int_{\mathbb{R}^d} \dot{f} dv$ by solving the free transport equation and integrating in v. This was used for example in the study of Landau damping by Mouhot and Villani [**36**]. Then by using Fourier analysis (in time and space) and assuming that $\overline{f}(v)$ satisfies a Penrose stability condition, one can derive relevant estimates from this integral equation and thus estimate $\dot{\rho}$ in $L^2_{t,x}$ with respect to the source term (without loss of derivatives).

Nevertheless, there are two major difficulties to overcome in order to apply this strategy in Sobolev spaces and in the general case where f depends also on t and x.

• The first one is due to subprincipal terms in (2.1). This comes from the fact that we do not expect that $2m$ derivatives of f can be controlled uniformly in ε (only $2m$ derivatives of ρ and $2m-1$ derivatives of f are). In (2.1), there are actually subprincipal terms under the form $\partial_x E \cdot \nabla_v \partial^{2m-1} f$ that involves $2m$ derivatives of f and thus cannot be considered as a remainder. The idea would be to replace the fields ∂_{x_i}, by more general vector fields in order to kill these subprincipal terms. However, since these fields have to depend on x, they do not commute with $v \cdot \nabla_x$ anymore and thus we would recreate other bad subprincipal terms. A way to overcome this issue consists in applying to the equation powers of some well-chosen second order differential operators designed to kill all bad subprincipal terms. This family of operators is introduced and studied in Lemma 5 and in Lemma 6.

In dimension one, the relevant operator is of the form

$$L := \partial_{xx} + \varphi \partial_x \partial_v + \psi \partial_{vv},$$

with (φ, ψ) satisfying the system

$$\begin{cases} (\partial_t + v \cdot \nabla_x + E \cdot \nabla_v)\varphi = \partial_x E + \text{ (zero order terms)}, & \varphi|_{t=0} = 0, \\ (\partial_t + v \cdot \nabla_x + E \cdot \nabla_v)\psi = \varphi \partial_x E + \text{ (zero order terms)}, & \psi|_{t=0} = 0, \end{cases}$$

this system being designed to kill the bad subprincipal term.

In dimension d, we obtain in a similar way some relevant operators $(L_{i,j})_{1 \le i,j \le d}$. By applying these operators to f, we obtain for all $I = (i_1, \cdots, i_m), J = (j_1, \cdots, j_m) \in \{1, \cdots, d\}^m$, the functions

$$f_{I,J} := L_{i_1,j_1} \cdots L_{i_m,j_m} f$$

that satisfy two key properties. First, they can be used to control ρ in the sense that

$$\int_{\mathbb{R}^d} f_{I,J}\, dv = \partial_x^{\alpha(I,J)} \rho + R,$$

where $\partial_x^{\alpha(I,J)}$ is of order $2m$ and R is a remainder (that is small and well controlled in small time), see Lemma 7. Furthermore, $f_{I,J}$ evolves according to the linearized equation about f at leading order, that is

$$\partial_t f_{I,J} + v \cdot \nabla_x f_{I,J} + E \cdot \nabla_v f_{I,J} + \partial_x^{\alpha(I,J)} E \cdot \nabla_v f + \cdots = 0,$$

where $\cdots$ is here a shorthand for lower order terms that we can indeed handle.

• Since f depends on x, there is a nontrivial electric field E in the above equation and we cannot derive an equation for the density just by inverting the free transport operator and by using Fourier analysis. We shall thus first make the change of variable $v \mapsto \Phi(t, x, v)$ in order to straighten the vector field

$$\partial_t + v \cdot \nabla_x + E \cdot \nabla_v \quad \text{into} \quad \partial_t + \Phi(t, x, v) \cdot \nabla_x,$$

where $\Phi(t, x, v)$ is the vector field satisfying the Burgers equation

$$\partial_t \Phi + \Phi \cdot \nabla_x \Phi = E, \quad \Phi|_{t=0} = v,$$

and is close to v in small time. By using the characteristics method, and another near identity change of variable, we can then obtain an integral equation for the evolution of $(\partial_x^{\alpha(I,J)} \rho)_{I,J}$ that has nice properties (see Lemma 13 and Lemma 14). For this stage, we need to study integral operators under the form

$$K_G(F)(t, x) = \int_0^t \int (\nabla_x F)(s, x - (t - s)v) \cdot G(t, s, x, v)\, dv ds.$$

Note that K_G seems to feature a loss of one derivative when acting on F, but we prove that it is actually a bounded operator on $L^2_{t,x}$, provided that $G(t, s, x, v)$ is smooth enough and localized in v, see Proposition 2. This is an effect in the same spirit as kinetic averaging lemmas (see e.g. [**13**]). We essentially end up with the study of the integral equation (with unknown $h(t, x)$)

$$h = K_{\nabla_v f^0} (I - \varepsilon^2 \Delta_x)^{-1}(h) + R,$$

where R is a remainder we can control.

• The last step of the proof consists in introducing a parameter $\gamma > 0$ (to be chosen large enough) and conjugating the integral equation by $e^{\gamma t}$, which leads to the study of the operator

$$\left(I - e^{-\gamma t} K_{\nabla_v f^0} (I - \varepsilon^2 \Delta_x)^{-1} e^{\gamma t}\right).$$

We finally relate this operator to a semiclassical pseudodifferential operator whose symbol is given by the Penrose function $\mathcal{P}(\gamma, \tau, \eta, f^0_\varepsilon(x, \cdot))$ (see Lemma 15). We refer to Section 8 for definitions and basic facts of pseudodifferential calculus.

We therefore observe that the Penrose stability condition (1.4) can be seen as a condition of *ellipticity* of this symbol. We can finally use a semi-classical pseudodifferential calculus (with parameter) in order to invert $\big(I - e^{-\gamma t} K_{\nabla_v f^0}(I - \varepsilon^2 \Delta_x)^{-1} e^{\gamma t}\big)$ up to a small remainder, which yields an estimate for $\partial_x^{2m} \rho$ in $L^2_{t,x}$, as achieved in Proposition 3. This is where choosing γ large is useful. Note that this part of the proof is very much inspired by the use of the Lopatinskii determinant in order to get estimates for initial boundary value problems for hyperbolic systems (see e.g. [**27, 34, 30**]) and the use of the Evans function in order to get estimates in singular limit problems involving stable boundary layers (see e.g. [**35, 17, 18, 38**]).

Combining all these a priori estimates, we end up with the inequality stated in (6.1), which gives a good control of $\mathcal{N}_{2m,2r}(T,f)$. It is then standard to conclude by a bootstrap argument, see Section 6.

In Section 7, we provide the proofs of Theorems 2 and 3. Theorem 2 follows from Theorem 1 and compactness arguments. Then, the existence part in Theorem 3 is a straightforward consequence of Theorem 2. The uniqueness part needs a specific analysis, which is performed in the same spirit as the way we obtained a priori estimates, see Proposition 4 and Corollary 2.

The last section of the paper is dedicated to some elements of pseudodifferential calculus which are needed in the proof.

3. Proof of Theorem 1: setting up the bootstrap argument

For the proof of the estimates that will eventually lead to the proof of Theorem 1, we shall systematically remove the subscripts ε for the solution f_ε of (1.1). The notation $A \lesssim B$ will stand as usual for $a \leq C\, a$ where C is a positive number that may change from line to line but which is independent of ε and of a, b. Similarly, Λ will stand for a continuous function which is independent of ε and which is non-decreasing with respect to each of its arguments.

3.1. Some useful Sobolev estimates. — Before starting the actual proof, let us state some basic product and commutator estimates that will be very useful in the paper. We denote by $[A, B] = AB - BA$ the commutator between two operators. We shall also use in the paper the notation $\|\cdot\|_{H^k_{x,v}}$ for the standard Sobolev norm on L^2 for functions depending on (x, v). In a similar way we will use the notations $\|\cdot\|_{W^{k,\infty}}$ and $\|\cdot\|_{W^{k,\infty}_{x,v}}$ for the standard Sobolev spaces on L^∞ for functions depending on x and (x, v) respectively.

Lemma 1. — *Let $s \geq 0$. Consider a smooth nonnegative function $\chi = \chi(v)$ that satisfies $|\partial^\alpha \chi| \leq C_\alpha \chi$ for every $\alpha \in \mathbb{N}^d$ such that $|\alpha| \leq s$.*

– *Consider two functions* $f = f(x, v)$, $g = g(x, v)$, *then we have for* $k \geq s/2$

$$\|\chi f g\|_{H^s_{x,v}} \lesssim \|f\|_{W^{k,\infty}_{x,v}} \|\chi g\|_{H^s_{x,v}} + \|g\|_{W^{k,\infty}_{x,v}} \|\chi f\|_{H^s_{x,v}}. \tag{3.1}$$

– *Consider a function* $E = E(x)$ *and a function* $F(x, v)$, *then we have for any* $s_0 > d$ *that*

$$\|\chi E F\|_{H^s_{x,v}} \lesssim \|E\|_{H^{s_0}_x} \|\chi F\|_{H^s_{x,v}} + \|E\|_{H^s_x} \|\chi F\|_{H^s_{x,v}}. \tag{3.2}$$

– *Consider a vector field* $E = E(x)$ *and a function* $f = f(x, v)$, *then we have for any* $s_0 > 1 + d$ *and for any* α, $\beta \in \mathbb{N}^d$ *such that* $|\alpha| + |\beta| = s \geq 1$ *that*

$$\left\|\chi \left[\partial_x^\alpha \partial_v^\beta, E(x) \cdot \nabla_v\right] f\right\|_{L^2_{x,v}} \lesssim \|E\|_{H^{s_0}_x} \|\chi f\|_{H^s_{x,v}} + \|E\|_{H^s_x} \|\chi f\|_{H^s_{x,v}}. \tag{3.3}$$

Note that by taking as weight function $\chi(v) = (1 + |v|^2)^{\pm \frac{\sigma}{2}}$, we can use this lemma to get estimates in $\mathcal{H}^s_{\pm\sigma}$. Note that (3.2), (3.1) are not sharp in terms of regularity but they will be sufficient for our purpose.

Proof of Lemma 1. — The estimate (3.1) is straightforward, using the pointwise estimates on χ and its derivatives. To prove (3.2), by using Leibnitz formula, we have to estimate

$$\|\chi \partial_x^\alpha E \partial_x^\beta \partial_v^\gamma F\|_{L^2_{x,v}}$$

with $|\alpha| + |\beta| + |\gamma| \leq s$. If $|\alpha| \leq d/2$, we write by Sobolev embedding in x that

$$\|\chi \partial_x^\alpha E \partial_x^\beta \partial_v^\gamma F\|_{L^2_{x,v}} \lesssim \|\partial_x^\alpha E\|_{L^\infty_x} \|\chi F\|_{H^s_{x,v}} \lesssim \|E\|_{H^{s_0}_x} \|\chi F\|_{H^s_{x,v}}.$$

If $|\alpha| > d/2$, by using again Sobolev embedding in x, we write

$$\|\chi \partial_x^\alpha E \partial_x^\beta \partial_v^\gamma F\|_{L^2_{x,v}} \lesssim \|E\|_{H^s} \Big(\int \sup_x |\chi \partial_x^\beta \partial_v^\gamma F|^2 \, dv \Big)^{\frac{1}{2}} \lesssim \|E\|_{H^s} \|\chi F\|_{H^s_{x,v}}$$

since $|\beta| + |\gamma| + \frac{d}{2} \leq s - |\alpha| + \frac{d}{2} < s$.

To prove (3.3), we proceed in a similar way. By expanding the commutator, we have to estimate

$$I_\gamma = \left\|\chi \partial_x^\gamma E \cdot \nabla_v \partial_x^{\alpha - \gamma} \partial_v^\beta f\right\|_{L^2_{x,v}}$$

for $0 < \gamma \leq \alpha$ where $|\alpha| + |\beta| = s$. If $0 < |\gamma| \leq 1 + \frac{d}{2}$, we write by using Sobolev embedding in x

$$I_\gamma \lesssim \|\partial_x^\gamma E\|_{L^\infty} \|\chi \nabla_v \partial_x^{\alpha-\gamma} \partial_v^\beta f\|_{L^2_{x,v}} \lesssim \|E\|_{H^{s_0}} \|\chi f\|_{H^s_{x,v}}.$$

If $|\gamma| > 1 + \frac{d}{2}$, we write by using again the Sobolev embedding in x

$$I_\gamma \lesssim \|\partial_x^\gamma E\|_{L^2} \left(\int \sup_x |\chi \nabla_v \partial_x^{\alpha-\gamma} \partial_v^\beta f|^2 \, dv \right)^{\frac{1}{2}} \lesssim \|E\|_{H^s} \|\chi f\|_{H^s_{x,v}}$$

since $1 + |\alpha| + |\beta| - |\gamma| = 1 + s - |\gamma| < s - \frac{d}{2}$.

□

We shall also use the following statement.

Lemma 2. — *For every $s \geq 0$, $\alpha, \beta \in \mathbb{N}^{2d}$ with $|\alpha| + |\beta| \leq s$, and $\chi(v)$ satisfying the assumptions of Lemma 1, we have for all functions $f = f(x,v)$, $g = g(x,v)$, the estimate*

$$\|\partial^\alpha_{x,v} f \, \partial^\beta_{x,v} g\|_{L^2} \lesssim \|\frac{1}{\chi} f\|_{L^\infty_{x,v}} \|\chi g\|_{H^s_{x,v}} + \|\chi g\|_{L^\infty_{x,v}} \|\frac{1}{\chi} f\|_{H^s_{x,v}}. \tag{3.4}$$

Proof of Lemma 2. — It suffices to notice that since χ and $1/\chi$ satisfy that $|\partial^\alpha \phi| \lesssim \phi$ for every $|\alpha| \leq s$, it is equivalent to estimate

$$\left\| \partial^{\tilde{\alpha}}_{x,v} \left(\frac{1}{\chi} f \right) \partial^{\tilde{\beta}}_{x,v} (\chi g) \right\|_{L^2}$$

with $\tilde{\alpha}$, $\tilde{\beta}$ that still satisfy $|\tilde{\alpha}| + |\tilde{\beta}| \leq s$ and the result follows from the standard tame Sobolev-Gagliardo-Nirenberg-Moser inequality. □

3.2. Set up of the bootstrap. — From classical energy estimates (that we shall recall below, see Proposition 1 and its proof), we easily get that the Vlasov-Poisson system is locally well-posed in $\mathcal{H}^{2m}_{2r}$ for any m and r satisfying $2m > 1 + d$ and $2r > d/2$. This means that if $f^0 \in \mathcal{H}^{2m}_{2r}$, there exists $T > 0$ (that depends on ε) and a unique solution $f \in \mathcal{C}([0,T], \mathcal{H}^{2m}_{2r})$ of the Vlasov-Poisson system (1.1). We can thus consider a maximal solution $f \in \mathcal{C}([0,T^*), \mathcal{H}^{2m}_{2r})$. Note that since $2r > d/2$, we have for every $T \in [0, T^*)$,

$$\|\rho\|_{L^2((0,T),H^{2m})} \lesssim T^{\frac{1}{2}} \sup_{[0,T]} \|f\|_{\mathcal{H}^{2m}_{2r}}. \tag{3.5}$$

and hence $\mathcal{N}_{2m\,2r}(T,f)$ (recall (1.8)) is well defined for $T < T^*$. From this local existence result, we can thus define another maximal time T^ε (that a priori depends on ε) as

$$T^\varepsilon = \sup \Big\{ T \in [0, T^*), \quad \mathcal{N}_{2m,2r}(T,f) \leq R \Big\}. \tag{3.6}$$

By taking R independent of ε but sufficiently large, we have by continuity that $T^\varepsilon > 0$. Our aim is to prove that R can be chosen large enough so that for all $\varepsilon \in (0,1]$, T^ε is uniformly bounded from below by some time $T > 0$. There are two possibilities for T^ε:

1. either $T^\varepsilon = T^*$,
2. or $T^\varepsilon < T^*$ and $\mathcal{N}_{2m,2r}(T^\varepsilon, f) = R$.

Let us first analyze the first case which is straightforward. If $T^\varepsilon = T^* = +\infty$, then the estimate $\mathcal{N}_{2m,2r}(T,f) \leq R$ holds for all times and there is nothing to do. We shall soon show that the scenario $T^\varepsilon = T^* < +\infty$ is impossible by using an energy estimate.

We shall denote by $\mathcal{T}$ the transport operator

$$\mathcal{T} := \partial_t + v \cdot \nabla_x + E \cdot \nabla_v, \tag{3.7}$$

where E is the electric field associated to f, that is $E = -\nabla(I - \varepsilon^2 \Delta)^{-1}(\int_{\mathbb{R}^d} f \, dv - 1)$.

We first write an identity (that follows from a direct computation) which we will use many times in this paper.

Lemma 3. — *For $\alpha, \beta \in \mathbb{N}^d$, we have for any smooth function f the formula*

$$\partial_x^\alpha \partial_v^\beta (\mathcal{T} f) = \mathcal{T}(\partial_x^\alpha \partial_v^\beta f) + \sum_{i=1}^d \mathbb{1}_{\beta_i \neq 0} \partial_{x_i} \partial_x^\alpha \partial_v^{\overline{\beta}^i} f + \left[\partial_x^\alpha \partial_v^\beta, E \cdot \nabla_v\right] f, \tag{3.8}$$

where $\overline{\beta}^i$ is equal to β except that $\overline{\beta}^i_i = \beta_i - 1$.

The $\mathcal{H}^{2m}_{2r}$ energy estimate reads as follows.

Proposition 1. — *For any solution f to* (1.1)*, we have, for some $C > 0$ independent of ε, the estimate*

$$\sup_{[0,T^\varepsilon)} \|f(t)\|^2_{\mathcal{H}^{2m}_{2r}} \leq \|f^0\|^2_{\mathcal{H}^{2m}_{2r}} \exp\left[C\left(T^\varepsilon + \frac{1}{\varepsilon}(T^\varepsilon)^{\frac{1}{2}} R\right)\right], \tag{3.9}$$

where T^ε and R are introduced in (3.6)*.*

Proof of Proposition 1. — For f satisfying (1.1) and thus $\mathcal{T} f = 0$, we can use the commutator formula (3.8), take the scalar product with $(1+|v|^2)^{2r}\partial_x^\alpha \partial_v^\beta f$, and sum for all $|\alpha| + |\beta| \leq 2m$. By using (3.3) with $s = 2m$, $\chi(v) = (1+|v|^2)^r$ and $s_0 = 2m$ (recall that $2m > 1 + d$), we get

$$\left\|\chi \left[\partial_x^\alpha \partial_v^\beta, E(x) \cdot \nabla_v\right] f\right\|_{L^2_{x,v}} \lesssim \|E\|_{H^{2m}} \|f\|_{\mathcal{H}^{2m}_{2r}}.$$

By Cauchy-Schwarz we thus have

$$\left|\int \chi \left[\partial_x^\alpha \partial_v^\beta, E(x) \cdot \nabla_v\right] f \;\; \chi \partial_x^\alpha \partial_v^\beta f\right| \lesssim \|E\|_{H^{2m}} \|f\|^2_{\mathcal{H}^{2m}_{2r}}.$$

We end up with a classical energy estimate

$$\frac{d}{dt}\|f\|^2_{\mathcal{H}^{2m}_{2r}} \lesssim \|f\|^2_{\mathcal{H}^{2m}_{2r}} + \|E\|_{H^{2m}} \|f\|^2_{\mathcal{H}^{2m}_{2r}}.$$

By using the elliptic regularity estimate for the Poisson equation which gives

$$\|E\|_{H^{2m}} = \|\nabla_x V\|_{H^{2m}} \lesssim \frac{1}{\varepsilon}\|\rho\|_{H^{2m}},$$

we get that for $t \in [0, T^\varepsilon)$ and for some $C > 0$ independent of ε,

$$\|f(t)\|^2_{\mathcal{H}^{2m}_{2r}} \leq \|f^0\|^2_{\mathcal{H}^{2m}_{2r}} + C\int_0^t \left(\frac{1}{\varepsilon}\|\rho\|_{H^{2m}} + 1\right) \|f(s)\|^2_{\mathcal{H}^{2m}_{2r}}$$

for some $C > 0$. Consequently, from the Gronwall inequality, we obtain that

$$\begin{aligned}\sup_{[0,T^\varepsilon)} \|f(t)\|^2_{\mathcal{H}^{2m}_{2r}} &\leq \|f^0\|^2_{\mathcal{H}^{2m}_{2r}} \exp\left[C\left(T^\varepsilon + \frac{1}{\varepsilon}(T^\varepsilon)^{\frac{1}{2}}\|\rho\|_{L^2([0,T^\varepsilon),H^{2m})}\right)\right] \\ &\leq \|f^0\|^2_{\mathcal{H}^{2m}_{2r}} \exp\left[C\left(T^\varepsilon + \frac{1}{\varepsilon}(T^\varepsilon)^{\frac{1}{2}}\mathcal{N}_{2m,\,2r}(T^\varepsilon, f)\right)\right],\end{aligned}$$

from which, since $\mathcal{N}_{2m,\,2r}(T^\varepsilon, f) \leq R$, we get the expected estimate.

□

In particular, if $T^\varepsilon = T^* < +\infty$, we have, according to Proposition 1,

$$\sup_{[0,T^*)} \|f(t)\|^2_{\mathcal{H}^{2m}_{2r}} \leq \|f^0\|^2_{\mathcal{H}^{2m}_{2r}} \exp\left[C\left(T^\varepsilon + \frac{1}{\varepsilon}(T^\varepsilon)^{\frac{1}{2}} R\right)\right] < +\infty.$$

This means that the solution could be continued beyond T^* and this contradicts the definition of T^*; as a consequence, this case is impossible.

Therefore, let us assume from now on that $T^\varepsilon < T^*$ and $\mathcal{N}_{2m,\,2r}(T^\varepsilon, f) = R$. We shall estimate $\mathcal{N}_{2m,\,2r}(T, f)$ and prove that for some well chosen parameter R (independent of ε), there exists some time $T^\# > 0$, small but independent of ε, such that the equality

$$\mathcal{N}_{2m,\,2r}(T, f) = R$$

cannot hold for any $T \in [0, T^\#]$. We will then deduce that $T^\varepsilon > T^\#$.

To this end, we need to estimate $\mathcal{N}_{2m,\,2r}(T, f)$. In order to estimate the part $\|f_\varepsilon\|_{L^\infty((0,T),\mathcal{H}^{2m-1}_{2r})}$ of the quantity, we can also proceed by using standard energy estimates. Then the main part of the work will be to control $\|\rho_\varepsilon\|_{L^2((0,T),H^{2m})}$ uniformly in ε by using the Penrose stability condition. Note that we cannot use the estimate (3.5) to get a control that is independent of ε since estimating $\|f\|_{\mathcal{H}^{2m}_{2r}}$ in terms of $\mathcal{N}_{2m,\,2r}(T, f)$ requires the use of the elliptic regularity provided by the Poisson equation, and thus costs negative powers of ε.

We end this section with the $\mathcal{H}^{2m-1}_{2r}$ energy estimate without loss in ε.

Lemma 4. — *For $2m > 2 + d$ and $2r > d/2$, we have for any solution f to* (1.1) *the estimate*

$$\sup_{[0,T]} \|f\|_{\mathcal{H}^{2m-1}_{2r}} \leq \|f^0\|_{\mathcal{H}^{2m-1}_{2r}} + T^{\frac{1}{2}} \Lambda(T, R), \tag{3.10}$$

for every $T \in [0, T^\varepsilon)$ where T^ε and R are introduced in (3.6).

Proof of Lemma 4. — Let $\alpha, \beta \in \mathbb{N}^d$ with $|\alpha| + |\beta| = 2m - 1$. We can use again the commutation formula (3.8) take the scalar product with $(1+|v|^2)^{2r}\partial_x^\alpha \partial_v^\beta f$, sum for all $|\alpha| + |\beta| \leq 2m - 1$ and use (3.3) with $s = 2m - 1$, $\chi(v) = (1 + |v|^2)^r$ and $s_0 = 2m - 1$ (which is licit since $2m > 2 + d$). We obtain that

$$\frac{d}{dt}\|f\|^2_{\mathcal{H}^{2m-1}_{2r}} \lesssim \|f\|^2_{\mathcal{H}^{2m-1}_{2r}} + \|E\|_{H^{2m-1}} \|f\|^2_{\mathcal{H}^{2m-1}_{2r}}. \tag{3.11}$$

Integrating in time we obtain that for every $T \in [0, T^\varepsilon]$, for some $C > 0$

$$\sup_{[0,T]} \|f\|_{\mathcal{H}^{2m-1}_{2r}} \leq \|f^0\|_{\mathcal{H}^{2m-1}_{2r}} + C \sup_{[0,T]} \|f\|_{\mathcal{H}^{2m-1}_{2r}} \left(T + \int_0^T \|E\|_{H^{2m-1}}\, dt\right).$$

By using Cauchy-Schwarz in time and the crude estimate

$$\|E\|_{H^{2m-1}} = \|\nabla_x V\|_{H^{2m-1}} \lesssim \|\rho\|_{H^{2m}} \tag{3.12}$$

which is uniform in ε since it does not use any elliptic regularity, we obtain that

$$\sup_{[0,T]} \|f\|_{\mathcal{H}^{2m-1}_{2r}} \leq \|f^0\|_{\mathcal{H}^{2m-1}_{2r}} + CR(T + T^{\frac{1}{2}} R),$$

since $\mathcal{N}_{2m,\,2r}(T^\varepsilon, f) \leq R$. This proves the estimate (3.10).

□

4. Proof of Theorem 1: preliminaries for the estimates on ρ

4.1. Definition of appropriate second order differential operators. — In order to estimate the H^{2m} norm of ρ, we need to introduce appropriate differential operators of order $2m$ which are well adapted to the Vlasov equation in the quasineutral scaling. The usual basic approach is to use the vector fields ∂_x, ∂_v and thus to apply ∂^α with $|\alpha| \leq 2m$ to the Vlasov equation. The hope is that up to harmless commutators, $\partial^\alpha f$ will evolve according to the linearized equation about f and thus that we will just have to understand the dynamics of this linearized equation. Nevertheless, there are unbounded terms arising because of commutators. The main problem is the subprincipal term $\partial E \cdot \nabla_v \partial^{2m-1} f$ that involves $2m$ derivatives of f and thus cannot be controlled by $\mathcal{N}_{2m,2r}(t, f)$ uniformly in ε. As already said before, we could try to use more complicated variable coefficients vector fields designed to kill this commutator term. But since these vector fields have to depend on x they would not commute any more with the free transport operator $v \cdot \nabla_x$ and thus we would recreate others bad subprincipal commutators. This heuristics motivates the analysis of this section. It turns out that the following second order operators (and their composition) are relevant since they have good commutation properties with the transport operator $\mathcal{T}$ (recall (3.7)).

Lemma 5. — *Let $(\varphi^{i,j}_{k,l},\, \psi^{i,j}_{k,l})_{i,j,k,l \in \{1,\cdots,d\}}$ be smooth solutions of the system:*

$$\begin{cases} \mathcal{T}\varphi^{i,j}_{k,l} = \psi^{i,j}_{k,l} + \psi^{i,j}_{l,k} - \displaystyle\sum_{1 \leq k',l' \leq d} \varphi^{i,j}_{k',l'} \varphi^{k',l'}_{k,l} + \delta_{k,j} \partial_{x_i} E_l + \delta_{k,i} \partial_{x_j} E_l, \\ \mathcal{T}\psi^{i,j}_{k,l} = - \displaystyle\sum_{1 \leq k',l' \leq d} \varphi^{i,j}_{k',l'} \psi^{k',l'}_{k,l} + \varphi^{i,j}_{k,l} \partial_{x_k} E_k, \end{cases} \tag{4.1}$$

where δ denotes the Kronecker function. We assume that for all k, l, the matrices $(\varphi^{i,j}_{k,l})_{1 \leq i,j \leq d}$ and $(\psi^{i,j}_{k,l})_{1 \leq i,j \leq d}$ are symmetric. Introduce the second order operators

$$L_{i,j} := \partial_{x_i x_j} + \sum_{1 \leq k,l \leq d} \left(\varphi^{i,j}_{k,l} \partial_{x_k} \partial_{v_l} + \psi^{i,j}_{k,l} \partial_{v_k v_l} \right), \quad i,j \in \{1, \cdots d\}. \tag{4.2}$$

Then for all smooth functions f, we have the formula

$$L_{i,j}\mathcal{T}(f) = \mathcal{T} L_{i,j}(f) + \partial_{x_i x_j} E \cdot \nabla_v f + \sum_{k,l} \varphi^{i,j}_{k,l} L_{k,l} f. \tag{4.3}$$

Proof of Lemma 5. — We have by direct computations

$$\partial_{x_i x_j}(\mathcal{T} f) = \mathcal{T}(\partial_{x_i x_j} f) + \partial_{x_i x_j} E \cdot \nabla_v f + \partial_{x_i} E \cdot \nabla_v \partial_{x_j} f + \partial_{x_j} E \cdot \nabla_v \partial_{x_i} f,$$

$$\varphi^{i,j}_{k,l} \partial_{x_k} \partial_{v_l}(\mathcal{T} f) = \mathcal{T}(\varphi^{i,j}_{k,l} \partial_{x_k} \partial_{v_l} f) + \varphi^{i,j}_{k,l} (\partial_{x_k x_l} f + \partial_{x_k} E \cdot \nabla_v \partial_{v_l} f) - \mathcal{T}(\varphi^{i,j}_{k,l}) \partial_{x_k} \partial_{v_l} f,$$

$$\psi^{i,j}_{k,l} \partial_{v_k v_l}(\mathcal{T} f) = \mathcal{T}(\psi^{i,j}_{k,l} \partial_{v_k v_l} f) + \psi^{i,j}_{k,l}(\partial_{v_k} \partial_{x_l} f + \partial_{v_l} \partial_{x_k} f) - \mathcal{T}(\psi^{i,j}_{k,l}) \partial_{v_k v_l} f.$$

We can rewrite

$$\varphi^{i,j}_{k,l} \partial_{x_k x_l} f = \varphi^{i,j}_{k,l} \left(L_{k,l} f - \sum_{k',l'} \left(\varphi^{k,l}_{k',l'} \partial_{x_{k'}} \partial_{v_{l'}} + \psi^{k,l}_{k',l'} \partial_{v_{k'} v_{l'}} \right) f \right),$$

which entails that

$$\begin{aligned} L_{i,j} \mathcal{T}(f) = {} & \mathcal{T} L_{i,j}(f) + \partial_{x_i x_j} E \cdot \nabla_v f + \sum_{k,l} \varphi^{i,j}_{k,l} L_{k,l} f \\ & + \sum_{k,l} \partial_{x_k} \partial_{v_l} f \left[-\mathcal{T} \varphi^{i,j}_{k,l} + \psi^{i,j}_{k,l} + \psi^{i,j}_{l,k} - \sum_{k',l'} \varphi^{i,j}_{k',l'} \varphi^{k',l'}_{k,l} + \delta_{k,j} \partial_{x_i} E_l + \delta_{k,i} \partial_{x_j} E_l \right] \\ & + \sum_{k,l} \partial_{v_k v_l} f \left[-\mathcal{T} \psi^{i,j}_{k,l} - \sum_{k',l'} \varphi^{i,j}_{k',l'} \psi^{k',l'}_{k,l} + \varphi^{i,j}_{k,l} \partial_{x_k} E_k \right]. \end{aligned}$$

We therefore deduce (4.3), because of (4.1). □

We shall now study the Sobolev regularity of the solution of the constraint equations (4.1).

Lemma 6. — *Assume $2m > 2 + d$ and $2r > d$. There exists $T_0 = T_0(R) > 0$ independent of ε such that for every $T < \min(T_0, T^\varepsilon)$, there exists a unique solution $(\varphi^{i,j}_{k,l}, \psi^{i,j}_{k,l})_{i,j,k,l}$ on $[0,T]$ of (4.1) satisfying*

$$\varphi^{i,j}_{k,l}|_{t=0} = \psi^{i,j}_{k,l}|_{t=0} = 0.$$

Moreover, we have the estimates

$$\sup_{[0,T]} \sup_{i,j,k,l} \|(\varphi^{i,j}_{k,l}, \psi^{i,j}_{k,l})\|_{W^{p,\infty}_{x,v}} \leq T^{\frac12} \Lambda(T,R), \quad p < 2m - d/2 - 2, \tag{4.4}$$

$$\sup_{[0,T]} \sup_{i,j,k,l} \|(\varphi^{i,j}_{k,l}, \psi^{i,j}_{k,l})\|_{\mathcal{H}^{2m-2}_{-r}} \leq T^{\frac12} \Lambda(T,R). \tag{4.5}$$

Finally, for all k,l, the matrices $(\varphi^{i,j}_{k,l})_{1\leq i,j\leq d}$ and $(\psi^{i,j}_{k,l})_{1\leq i,j\leq d}$ are symmetric.

Proof of Lemma 6. — System (4.1) is a system of semi-linear transport equations that is coupled only via a zero order term. The existence and uniqueness of a smooth solution can be obtained by a standard fixed point argument and thus we shall only focus on a priori estimates. Note that the symmetry of the matrices $(\varphi^{i,j}_{k,l})_{1\leq i,j\leq d}$ and $(\psi^{i,j}_{k,l})_{1\leq i,j\leq d}$ for all k,l is a consequence of the uniqueness of the solution.

Let us set

$$M_p(T) = \sup_{[0,T]} \sup_{i,j,k,l} \|(\varphi^{i,j}_{k,l}(t), \psi^{i,j}_{k,l}(t))\|_{W^{p,\infty}_{x,v}}.$$

We first apply $\partial_x^\alpha \partial_v^\beta$ for $|\alpha| + |\beta| \le p$ to (4.1) and use again the commutation formula (3.8). By using the maximum principle for the transport operator $\mathcal{T}$, we get that

$$M_p(T) \lesssim T\,(1 + M_p(T))\, M_p(T) + (1 + M_p(T)) \int_0^T \|\nabla E\|_{W^{p,\infty}}\, dt.$$

By Sobolev embedding and (3.12), we obtain

$$\int_0^T \|\nabla E\|_{W^{p,\infty}}\, dt \lesssim \int_0^T \|E\|_{H^{2m-1}}\, dt \lesssim T^{\frac12} R$$

since $1 + p + \frac{d}{2} < 2m - 1$ and hence we find

$$M_p(T) \lesssim T^{\frac12} R + (T + T^{\frac12} R) M_p(T) + M_p(T)^2.$$

Consequently, there exists $\gamma_0 > 0$ independent of ε but sufficiently small such that for every $T \le T_0$ satisfying $(T_0 + T_0^{\frac12} R) \le \gamma_0$, we get the estimate

$$M_p(T) \le T^{\frac12} R.$$

For the estimate (4.5), we apply $\partial_x^\alpha \partial_v^\beta$ with $|\alpha| + |\beta| \le 2m - 2$ to the system (4.1), we use the commutation formula (3.8) with the transport operator, multiply by the weight $(1 + |v|^2)^{-r}$ and take the scalar product by $\partial_x^\alpha \partial_v^\beta (\varphi^{i,j}_{k,l}, \psi^{i,j}_{k,l})$ as usual. Let us set

$$Q_{2m-2}(t) = \left(\sum_{i,j,k,l} \|(\varphi^{i,j}_{k,l}, \psi^{i,j}_{k,l})(t)\|^2_{\mathcal{H}^{2m-2}_{-r}} \right)^{\frac12}.$$

By using

- the product estimate (3.1) with $s = 2m - 2$, $k = s/2 = m - 1$, to handle the quadratic terms in the right hand side,
- the commutator estimate (3.3) with $s = 2m - 2$, $s_0 = 2m - 1\,(> 1 + d)$, to handle the commutators with E,

we obtain for $t \in [0, T]$, $T < \min(T_0, T^\varepsilon)$ that

$$\frac{d}{dt} \|(\varphi^{i,j}_{k,l}, \psi^{i,j}_{k,l})\|^2_{\mathcal{H}^{2m-2}_{-r}} \lesssim (1 + M_{m-1}(T) + \sup_{[0,T]} \|E\|_{H^{2m-1}}) Q_{2m-2}(t)^2 + \left\| \frac{1}{(1 + |v|^2)^{\frac{r}{2}}} E \right\|_{H^{2m-1}_{x,v}} \|(\varphi^{i,j}_{k,l}, \psi^{i,j}_{k,l})\|_{\mathcal{H}^{2m-2}_{-r}}.$$

Note that since $2r > d$, we have that

$$\left\| \frac{1}{(1 + |v|^2)^{\frac{r}{2}}} E \right\|_{H^{2m-1}_{x,v}} \lesssim \|E\|_{H^{2m-1}_x}.$$

Consequently, we can sum over i, j, k, l, and use (3.12) and the estimate (4.4) (since $m-1 < 2m - \frac{d}{2} - 2$ is equivalent to $2m > 2 + d$), to obtain that

$$\frac{d}{dt} Q_{2m-2}(t) \lesssim (1 + T^{\frac{1}{2}} \Lambda(T, R)) Q_{2m-2}(t) + \|\rho\|_{H^{2m}}.$$

The estimate (4.5) thus follows from the Gronwall inequality and Cauchy-Schwarz.

□

We now study the effect of composing $L_{i,j}$ operators. In the rest of this paper, we shall use upper case letters (like I, J) for multi-indices in $\{1, \cdots, d\}^m$, as opposed to lower case letters (like i, j) for indices in $\{1, \cdots, d\}$.

Lemma 7. — *Assume $2m > 3 + d$ and $2r > d$. For $I = (i_1, \cdots, i_m), J = (j_1, \cdots, j_m) \in \{1, \cdots, d\}^m$, define*

$$f_{I,J} = L^{(I,J)} f := L_{i_1,j_1} \cdots L_{i_m,j_m} f \tag{4.6}$$

For every $T < \min(T_0, T^\varepsilon)$, we first have that for $I, J \in \{1, \cdots, d\}^m$

$$\int_{\mathbb{R}^d} f_{I,J}\, dv = \int_{\mathbb{R}^d} \partial_x^{\alpha(I,J)} f\, dv + \mathcal{R} = \partial_x^{\alpha(I,J)} \rho + \mathcal{R}, \tag{4.7}$$

with $\alpha(I, J) = (\alpha_k(I, J))_{1 \le k \le d}$, $\alpha_k(I, J)$ being equal to the number of occurrences of k in the set $\{i_1, \cdots, i_m, j_1, \cdots, j_m\}$ and $\mathcal{R}$ is a remainder satisfying

$$\|\mathcal{R}\|_{L^\infty((0,T), L^2_x)} \lesssim \Lambda(T, R). \tag{4.8}$$

Moreover, for f satisfying (1.1)*, we have that $f_{I,J}$ solves*

$$\mathcal{T}(f_{I,J}) + \partial_x^{\alpha(I,J)} E \cdot \nabla_v f + \mathcal{M}_{I,J} \mathcal{F} = F_{I,J}, \tag{4.9}$$

where

$$\mathcal{F} := (f_{I,J})_{I,J \in \{1,\cdots,d\}^m}, \quad \mathcal{M}_{I,J} \mathcal{F} := \sum_{k=1}^{m} \sum_{1 \le k', l' \le d} \varphi_{k',l'}^{i_k, j_k} f_{I_{k \to k'}, J_{k \to l'}}, \tag{4.10}$$

where $I_{s \to t}$ denotes the element of $\{1, \cdots, d\}^m$ which is equal to I except for its s-th element which is equal to t and $F = (F_{I,J})$ is a remainder satisfying

$$\|F\|_{L^2((0,T), \mathcal{H}^0_r)} \le \Lambda(T, R), \quad \forall T < \min(T_0, T^\varepsilon). \tag{4.11}$$

Proof of Lemma 7. — At first, we can expand $f_{I,J} = L_{i_1,j_1} \cdots L_{i_m,j_m} f$ in a more tractable form. Let us set $U = (\varphi_{k,l}^{i_\alpha, j_\beta}, \psi_{k,l}^{i_\alpha, j_\beta})_{1 \le k,l \le d,\, 1 \le \alpha, \beta \le m}$. Then, we can write

$$\begin{aligned} f_{I,J} &= \partial_x^{\alpha(I,J)} f + \sum_{s=0}^{2m-2} \sum_{e,\, \alpha,\, k_0 \cdots, k_s} P_{s,e,\alpha}^{k_0}(U) P_{s,e,\alpha}^{k_1}(\partial U) \cdots P_{s,e,\alpha}^{k_s}(\partial^s U) \partial_v^e \partial^\alpha f \\ &=: \partial_x^{\alpha(I,J)} f + \sum_{s=0}^{2m-2} \sum_{e,\, \alpha,\, k_0 \cdots k_s} \mathcal{R}_{s,e,\alpha}^{k_0, \cdots k_s}, \end{aligned} \tag{4.12}$$

where the sum is taken on indices such that

$$|e| = 1,\ |\alpha| = 2m - 1 - s,\ k_0 + k_1 + \cdots k_s \le m,\ k_0 \ge 1,\ k_1 + 2k_2 + \cdots s k_s = s, \tag{4.13}$$

and for all $0 \leq p \leq s$, $P^{k_p}_{s,e,\alpha}(X)$ is a polynomial of degree smaller than k_p (we denote by $\partial^k U$ the vector made of all the partial derivatives of length k of all components of U). The existence of an expansion under this form can be easily proven by induction. We can set

$$\mathcal{R} = \int_{\mathbb{R}^d} \sum_{s=0}^{2m-2} \sum_{e,\,\alpha,\,k_0,\cdots,k_s} \mathcal{R}^{k_0,\cdots,k_s}_{s,e,\alpha}\, dv,$$

so that we have to estimate $\int_{\mathbb{R}^d} \mathcal{R}^{k_0,\cdots,k_s}_{s,e,\alpha}\, dv$. All the following estimates are uniform in time for $t \in [0,T]$ with $T \leq \min(T_0, T^\varepsilon)$ but we do not mention the time parameter for notational convenience.

Let us start with the case where $\mathcal{R}^{k_0,\cdots k_s}_{e,s,\alpha}$ contains the maximal number of derivatives applied to f that is to say when $|\alpha| = 2m-1$ so that $2m$ derivatives of f are involved. In this case, we have $s = 0$ and hence

$$\int_{\mathbb{R}^d} \mathcal{R}^{k_0}_{e,0,\alpha}\, dv = \int_{\mathbb{R}^d} P^{k_0}_{s,e,\alpha}(U) \partial_v \partial^\alpha f\, dv,$$

where k_0 is less than m. We can thus integrate by parts in v to obtain that

$$\left\| \int_{\mathbb{R}^d} \mathcal{R}^{k_0}_{e,0,\alpha}\, dv \right\|_{L^2_x} \lesssim \Lambda(\|U\|_{W^{1,\infty}_{x,v}}) \left\| \int_{\mathbb{R}^d} |\partial^\alpha f|\, dv \right\|_{L^2_x} \lesssim \Lambda(\|U\|_{W^{1,\infty}_{x,v}}) \|f\|_{\mathcal{H}^{2m-1}_r}.$$

Consequently, by using (4.4) (since by assumption on m, we have $1 < 2m - 2 - \frac{d}{2}$) and (3.10) (since $2r > d$), we obtain that on $[0,T]$,

$$\left\| \int_{\mathbb{R}^d} \mathcal{R}^{k_0}_{e,0,\alpha}\, dv \right\|_{L^2_x} \leq \Lambda(T,R).$$

It remains to estimate the terms for which $s \geq 1$. Note that for all these terms the total number of derivatives applied to f is at most $2m-1$.

- When $s < 2m - \frac{d}{2} - 2$, we can use (4.4) to obtain that

$$\|P^{k_0}_{s,e,\alpha}(U) P^{k_1}_{s,e,\alpha}(\partial U) \cdots P^{k_s}_{s,e,\alpha}(\partial^s U)\|_{L^\infty_{x,v}} \leq \Lambda(T,R)$$

and hence we obtain as above that

$$\left\| \int \mathcal{R}^{k_0,\cdots,k_s}_{e,s,\alpha}\, dv \right\|_{L^2_x} \leq \Lambda(T,R) \|f\|_{\mathcal{H}^{2m-1}_r} \leq \Lambda(T,R).$$

- Let us now consider $s \geq 2m - 2 - \frac{d}{2}$. Let us start with the case where in the sequence $(k_1, \cdots, k_s)$, the bigger index l such that $k_l \neq 0$ and $k_p = 0$ for every $p > l$ is such that $l > s/2$. In this case, since $l k_l \leq s$, we necessarily have $k_l = 1$. Moreover, for the indices $p < l$ such that $k_p \neq 0$, we must have $p \leq p k_p < s/2$. Thus, we can use (4.4) to estimate $\|\partial^p U\|_{L^\infty_{x,v}}$ provided $s/2 \leq 2m - \frac{d}{2} - 2$. Since $s \leq 2m-2$, this is verified thanks to the assumption that $2m > 2 + d$. We thus obtain that

$$\left\| \int \mathcal{R}^{k_0,\cdots k_s}_{e,s,\alpha}\, dv \right\|_{L^2_x} \leq \Lambda(T,R) \left\| \int \partial^l U \partial^e_v \partial^\alpha f\, dv \right\|_{L^2_x}.$$

Next, we can use that

$$\begin{aligned}\left\|\int \partial^l U \partial_v^e \partial^\alpha f\right\|_{L^2_x} &\lesssim \left\| \|\frac{1}{(1+|v|^2)^{\frac{r}{2}}}\partial^l U\|_{L^2_v} \|(1+|v|^2)^{\frac{r}{2}}\partial_v^e\partial^\alpha f\|_{L^2_v}\right\|_{L^2_x} \\ &\lesssim \|U\|_{\mathcal{H}^{2m-2}_{-r}} \sup_x \|(1+|v|^2)^{\frac{r}{2}}\partial_v^e\partial^\alpha f\|_{L^2_v}.\end{aligned}$$

By Sobolev embedding in x, we have

$$\sup_x \|(1+|v|^2)^{\frac{r}{2}}\partial_v^e\partial^\alpha f\|_{L^2_v} \lesssim \|f\|_{\mathcal{H}^{2m-1}_r}$$

as soon as $2m-1 > 1+|\alpha|+\frac{d}{2} = 1+2m-1-s+\frac{d}{2}$ which is equivalent to $s > 1+\frac{d}{2}$. Since we are in the case where $s \geq 2m-2-\frac{d}{2}$, the condition is matched since $2m > 3+d$. Consequently, by using (3.10) and (4.5), we obtain again that

$$\left\|\int \mathcal{R}^{k_0,\cdots k_s}_{e,s,\alpha}\, dv\right\|_{L^2_x} \lesssim \Lambda(T,R).$$

Finally, it remains to handle the case where $k_l = 0$ for every $l > s/2$. Due to the assumption that $2m > 3+d$, we have by the same argument as above that since $s \leq 2m-2$, we necessarily have $\frac{s}{2} < 2m - \frac{d}{2} - 2$ and hence by using again (4.4) we find

$$\|\partial^l U\|_{L^\infty_{x,v}} \leq \Lambda(T,R), \quad l \leq s/2.$$

We deduce

$$\left\|\int \mathcal{R}^{k_0,\cdots k_s}_{e,s,\alpha}\, dv\right\|_{L^2_x} \leq \Lambda(T,R)\|f\|_{\mathcal{H}^{2m-1}_r} \leq \Lambda(T,R).$$

This ends the proof of (4.8).

To prove (4.9), (4.11), we apply $L^{(I,J)}$ to (1.1) and use the identity (4.3). We get for $m \geq 2$, the expression for the source term $F_{I,J}$

$$F_{I,J} = -\left(F_1 + F_2 + F_3 + F_4\right) \tag{4.14}$$

where

(4.15)
$$F_1 = \sum_{k=2}^{m-1} L_{i_1,j_1} \cdots L_{i_{m-k},j_{m-k}}((\partial^2_{x_{i_{m-k+1}},x_{j_{m-k+1}}} E) \cdot \nabla_v L_{i_{m-k+2},j_{m-k+2}} \cdots L_{i_m,j_m} f),$$

(4.16)
$$F_2 = L_{i_1,j_1} \cdots L_{i_{m-1},j_{m-1}} \left(\partial^2_{x_{i_m},x_{j_m}} E \cdot \nabla_v f\right) - \partial_x^{\alpha(I,J)} E \cdot \nabla_v f,$$

(4.17)
$$F_3 = \sum_{k=2}^{m-1} L_{i_1,j_1} \cdots L_{i_{m-k},j_{m-k}} \left(\sum_{k',l'} \varphi_{k',l'}^{i_{m-k+1},j_{m-k+1}} L_{k',l'} L_{i_{m-k+2},j_{m-k+2}} \cdots L_{i_m,j_m} f \right)$$
$$- \sum_{k=2}^{m-1} \sum_{k',l'} \varphi_{k',l'}^{i_{m-k+1},j_{m-k+1}} L^{(I_{m-k+1\to k'},J_{m-k+1\to l'})} f,$$

(4.18)
$$F_4 = L_{i_1,j_1} \cdots L_{i_{m-1},j_{m-1}} \left(\sum_{k,l} \varphi_{k,l}^{i_m,j_m} L_{k,l} f \right) - \sum_{k,l} \varphi_{k,l}^{i_m,j_m} L^{(I_{m\to k},J_{m\to l})}.$$

Estimate of $\mathbf{F_1}$. We shall first study the estimate for F_1. We have to estimate terms under the form

$$F_{1,k} = L^{m-k} G_k, \quad G_k = \partial^2 E \cdot \nabla_v L^{k-1} \tag{4.19}$$

where we use the notation L^n for the composition of n $L_{i,j}$ operators (the exact combination of the operators involved in the composition does not matter). Note that as in (4.12), we can develop L^n under the form

$$L^n = \partial_x^{\alpha_n} + \sum_{s=0}^{2n-2} \sum_{e,\,\alpha,\,k_0\cdots k_s} P^{k_0}_{s,e,\alpha}(U) P^{k_1}_{s,e,\alpha}(\partial U) \cdots P^{k_s}_{s,e,\alpha}(\partial^s U) \partial_v^e \partial^\alpha, \tag{4.20}$$

where for all $0 \leq p \leq s$, $P^{k_p}_{s,e,\alpha}(X)$ is a polynomial of degree smaller than k_p, the multi-index α_n has length $2n$ and the sum is taken on indices such that

$$|e| = 1,\ |\alpha| = 2n-1-s,\ k_0 + k_1 + \cdots k_s \leq n,\ k_0 \geq 1,\ k_1 + 2k_2 + \cdots s k_s = s. \tag{4.21}$$

Let us first establish a general useful estimate. We set for any fonction $G(x,v)$,

$$J_p(G)(x,v) = \sum_{s,\,\beta,K\in E} J_{p,s,\beta,K}(G) \tag{4.22}$$

where $K = (k_0, \cdots, k_s)$ and

$$J_{p,s,\beta,K}(G)(x,v) = P^{k_0}_{s,\beta}(U) P^{k_1}_{s,\beta}(\partial U) \cdots P^{k_s}_{s,\beta}(\partial^s U) \partial^\beta G$$

where for all $0 \leq p \leq s$, $P^{k_p}_{s,\beta}(X)$ is a polynomial of degree smaller than k_p and the sum is taken over indices belonging to the set E defined by

$$|\beta| = p - s,\ k_0 + k_1 + \cdots k_s \leq p/2,\ k_1 + 2k_2 + \cdots sk_s = s,\ 0 \leq s \leq p-2. \tag{4.23}$$

Lemma 8. — *For $2m-1 \geq p \geq 1$, $2m > d+3$, $2r > d$ and s, p, K satisfying* (4.23), *we have the estimate*

$$\|J_p(G)\|_{\mathcal{H}^0_r} \leq \Lambda(T,R)\Big(\|G\|_{\mathcal{H}^p_r} + \sum_{\substack{l \geq 2m - \frac{d}{2} - 2, \\ l + |\alpha| \leq p,\, |\alpha| \geq 2}} \|\partial^l U \partial^\alpha G\|_{\mathcal{H}^0_r}\Big). \tag{4.24}$$

Proof of Lemma 8. — For the terms in the sum such that $s < 2m - \frac{d}{2} - 2$, we can use (4.4) to obtain that

$$\|J_{p,s,\beta,K}(G)\|_{\mathcal{H}^0_r} \leq \Lambda(T,R)\|G\|_{\mathcal{H}^p_r}.$$

When $s \geq 2m - \frac{d}{2} - 2$, we first consider the terms for which in the sequence $(k_1, \cdots k_s)$ the biggest index l for which $k_l \neq 0$ is such that $l < 2m - \frac{d}{2} - 2$. Then again thanks to (4.4), we obtain that

$$\|J_{p,s,\beta,K}(G)\|_{\mathcal{H}^0_r} \leq \Lambda(T,R)\|G\|_{\mathcal{H}^p_r}.$$

When $l \geq 2m - \frac{d}{2} - 2$, we first observe that we necessarily have $k_l = 1$. Indeed if $k_l \geq 2$, because of (4.23), we must have $l \leq \frac{s}{2}$. This is possible only if $2m - \frac{d}{2} - 2 \leq \frac{p-2}{2} \leq \frac{2m-3}{2}$ that is to say $m \leq \frac{d}{2} + 1$ and hence this is impossible. Consequently $k_l = 1$. Moreover we note that for the other indices $\tilde{l}$ for which $k_{\tilde{l}} \neq 0$, because of (4.23), we must have $\tilde{l} k_{\tilde{l}} \leq s - l k_l$, so that

$$\tilde{l} \leq s - l \leq s - 2m + \frac{d}{2} + 2 \leq \frac{d}{2} - 1$$

and we observe that $\frac{d}{2} - 1 < 2m - \frac{d}{2} - 2$. Consequently, by another use of (4.4), we obtain that

$$\|J_{p,s,\beta,K}(G)\|_{\mathcal{H}^0_r} \leq \Lambda(T,R) \sum_{\substack{l \geq 2m - \frac{d}{2} - 2, \\ l + |\alpha| \leq p,\, |\alpha| \geq 2}} \|\partial^l U \partial^\alpha G\|_{\mathcal{H}^0_r}.$$

The fact that $|\alpha| \geq 2$ comes from (4.23). This ends the proof of Lemma 8. □

We shall now estimate $F_{1,k}$. Let us start with the case where $k \geq m/2$. Looking at the expansion of L^{m-k} given by (4.20), we have to estimate terms under the form $J_{2p}(G_k)$ for $2p \leq 2(m-k) \leq m$. We can thus use Lemma 8. Moreover, we observe that in the right hand side of (4.24), we have that $l \leq 2(m-k) - 2 \leq m - 2$, consequently, by assumption on m, we have $l < 2m - \frac{d}{2} - 2$ and hence we can estimate $\|\partial^l U\|_{L^\infty}$ by using (4.4). This yields

$$\|F_{1,k}\| \leq \Lambda(T,R)\|G_k\|_{\mathcal{H}^{2(m-k)}_r}, \quad k \geq m/2.$$

Next, we use (3.2) with $s = 2(m-k)$ and $s_0 = 2m-3\,(> d)$, and the definition of G_k in (4.19) to estimate the above right hand side. Since $d+2 < 2m-1$ by assumption on m and $2(m-k)+2 \leq 2m-1$ (since $k \geq 2$), we obtain

(4.25)

$$\begin{aligned}&\|F_{1,k}\|_{\mathcal{H}^0_r}\\&\leq \Lambda(T,R)\big(\|E\|_{H^{2m-1}}\|\nabla_v L^{k-1} f\|_{\mathcal{H}^{2(m-k)}_r} + \|E\|_{H^{2(m-k)+2}}\|\nabla_v L^{k-1} f\|_{\mathcal{H}^{2(m-k)}_r}\big)\\&\leq \Lambda(T,R)\|E\|_{H^{2m-1}}\|\nabla_v L^{k-1} f\|_{\mathcal{H}^{2(m-k)}_r}.\end{aligned}$$

By using again (3.12), this yields

$$\|F_{1,k}\|_{L^2([0,T],\mathcal{H}^0_r)} \leq \Lambda(T,R)\|\nabla_v L^{k-1} f\|_{L^\infty([0,T],\mathcal{H}^{2(m-k)}_r)}.$$

To estimate the above right hand side, we need to estimate $\partial^\gamma_{x,v} L^{k-1} f$ with $|\gamma| \leq 2m-2k+1$. By taking derivatives using the expression (4.20), we see that we have to estimate terms under the form $J_p(f)$ with $p \leq 2m-1$. Using Lemma 8, we thus obtain that

$$\begin{aligned}&\|F_{1,k}\|_{L^2([0,T],\mathcal{H}^0_r)}\\&\quad\leq \Lambda(T,R)\Big(\|f\|_{L^\infty([0,T],\mathcal{H}^{2m-1}_r)} + \sum_{\substack{l \geq 2m-\frac{d}{2}-2,\\ l+|\alpha|\leq 2m-1,\,|\alpha|\geq 2}} \|\partial^l U \partial^\alpha f\|_{L^\infty([0,T],\mathcal{H}^0_r)}\Big).\end{aligned}$$

To estimate the right hand side, since $|\alpha| \geq 2$ and $|\alpha|-2+l \leq 2m-3$, we can use (3.4), to obtain that

$$\|\partial^l U (1+|v|^2)^{\frac{r}{2}} \partial^\alpha f\|_{L^2_{x,v}} \lesssim \|U\|_{\mathcal{H}^{2m-3}_{-r}}\|(1+|v|^2)^r \partial^2 f\|_{L^\infty} + \|U\|_{L^\infty}\|f\|_{\mathcal{H}^{2m-1}_{2r}}.$$

By using again (4.5), (4.4) and the Sobolev embedding, we finally obtain that

$$\|F_{1,k}\|_{L^2([0,T],\mathcal{H}^0_r)} \leq \Lambda(T,R)\|f\|_{L^\infty([0,T],\mathcal{H}^{2m-1}_{2r})} \leq \Lambda(T,R), \quad k \geq m/2. \tag{4.26}$$

It remains to handle the case $k \leq m/2$. Again, by using (4.19) and the expansion (4.20), we first have to estimate terms under the form $J_{2(m-k)}(G_k)$. By using again Lemma 8, we first obtain

$$\begin{aligned}&\|F_{1,k}\|_{L^2([0,T],\mathcal{H}^0_r)}\\&\leq \Lambda(T,R)\Big(\|G_k\|_{L^2([0,T],\mathcal{H}^{2(m-k)}_r)} + \sum_{\substack{l \geq 2m-\frac{d}{2}-2,\\ l+|\alpha|\leq 2(m-k),\,|\alpha|\geq 2}} \|\partial^l U \partial^\alpha G_k\|_{L^2([0,T],\mathcal{H}^0_r)}\Big).\end{aligned}$$

By using the expression in (4.19) for G_k, we have to estimate terms under the form

$$\|\partial^l U \partial^\beta \partial^2 E\, \partial^\gamma \nabla_v L^{k-1} f\|_{\mathcal{H}^0_r}$$

with $l \geq 2m - \frac{d}{2} - 2$ and $l + |\beta| + |\gamma| \leq 2(m-k)$. Note that this implies that $|\beta| \leq 2(m-k) - l \leq \frac{d}{2} + 2 - 2k \leq \frac{d}{2} - 2$ since we have $k \geq 2$ (see (4.15)). In particular this yields $|\beta|+2+\frac{d}{2}<2m-2$ and thus by using again the Sobolev embedding (in x) and (3.12) we obtain that

$$\begin{aligned}\|\partial^l U \partial^\beta \partial^2 E \partial^\gamma \nabla_v L^{k-1} f\|_{\mathcal{H}^0_r} &\lesssim \|\rho\|_{H^{2m-1}} \|\partial^l U \partial^\gamma \nabla_v L^{k-1} f\|_{\mathcal{H}^0_r} \\ &\lesssim \|f\|_{\mathcal{H}^{2m-1}_r} \|\partial^l U \partial^\gamma \nabla_v L^{k-1} f\|_{\mathcal{H}^0_r} \\ &\leq \Lambda(T,R) \|\partial^l U \partial^\gamma \nabla_v L^{k-1} f\|_{\mathcal{H}^0_r}.\end{aligned}$$

Thus it remains to estimate $\|\partial^l U \partial^\gamma \nabla_v L^{k-1} f\|_{\mathcal{H}^0_r}$ for $l \geq 2m-\frac{d}{2}-2$ and $l+|\gamma| \leq 2(m-k)$. By using again (4.20), we can expand $\partial^\gamma \nabla_v L^{k-1} f$ as an expression under the form $J_{2k+|\gamma|-1}(f)$. Since we have that $2k+|\gamma|-1 \leq 1+\frac{d}{2} < 2m-\frac{d}{2}-2$, we can use (4.4) again to estimate all the terms involving U and its derivatives in in L^∞, this yields

$$\|\partial^l U \partial^\gamma \nabla_v L^{k-1} f\|_{\mathcal{H}^0_r} \leq \Lambda(T,R) \sum_{\tilde{\gamma}} \|\partial^l U \partial^{\tilde{\gamma}} f\|_{\mathcal{H}^0_r}$$

with $|\tilde{\gamma}| \leq |\gamma|+2k-1$ and thus $l+|\tilde{\gamma}| \leq 2m-1$ and $|\tilde{\gamma}| \geq 2$ (since $k \geq 2$). Consequently, by using again (3.4), we obtain that

$$\|\partial^l U \partial^\gamma \nabla_v L^{k-1} f\|_{\mathcal{H}^0_r} \leq \Lambda(T,R) \left(\|U\|_{L^\infty} \|f\|_{\mathcal{H}^{2m-1}_{2r}} + \|(1+|v|^2)^r \partial^2 f\|_{L^\infty_{x,v}} \|U\|_{\mathcal{H}^{2m-3}_{-r}} \right)$$

and we conclude finally by using (4.4), (4.5) and the Sobolev embedding that

$$\|F_{1,k}\|_{L^2([0,T],\mathcal{H}^0_r)} \leq \Lambda(T,R), \quad k \leq m/2 \tag{4.27}$$

(actually for this case we even have a slightly better L^∞ in time estimate). Looking at (4.26), (4.27), we have thus proven that

$$\|F_1\|_{L^2([0,T],\mathcal{H}^0_r)} \leq \Lambda(T,R). \tag{4.28}$$

Estimate of $\mathbf{F_2}$. We shall now turn to the study of F_2. By using (4.20) again, we can expand F_2 under the form

$$F_2 = \mathcal{C} + \sum_{|e|=1} J_{2m-3}(\partial_v^e(\partial^2 E \cdot \nabla_v f)),$$

where J_{2m-3} was defined in (4.22) and with

$$\mathcal{C} = \partial_x^{\alpha(\tilde{I},\tilde{J})} \left(\partial^2_{x_{i_m},x_{j_m}} E \cdot \nabla_v f \right) - \partial_x^{\alpha(I,J)} E \cdot \nabla_v f = [\partial_x^{\alpha(\tilde{I},\tilde{J})}, \nabla_v f] \cdot \partial^2_{x_{i_m},x_{j_m}} E.$$

where $\tilde{I} = (i_1, \cdots, i_{m-1}), \tilde{J} = (j_1, \cdots, j_{m-1})$ and thus $\alpha(\tilde{I},\tilde{J})$ has length $2m-2$. Note that this time, we have really used that in the expansion (4.20), the terms in the sum always involve at least one v derivative. By using Lemma 8, we get that

$$\begin{aligned}\|J_{2m-3}(\partial_v^e(\partial^2 E \cdot \nabla_v f))\|_{\mathcal{H}^0_r} \leq \Lambda(T,R) \Big(&\|\partial_v^e(\partial^2 E \cdot \nabla_v f)\|_{\mathcal{H}^{2m-3}_r} \\ &+ \sum_{\substack{l \geq 2m-\frac{d}{2}-2, \\ l+|\alpha| \leq 2m-3,\, |\alpha| \geq 2}} \|\partial^l U \partial^\alpha \partial_v^e(\partial^2 E \cdot \nabla_v f)\|_{\mathcal{H}^0_r} \Big).\end{aligned}$$

To estimate the first term, we can use (3.2) and (3.12) to obtain that

$$\|\partial_v(\partial^2 E \cdot \nabla_v f)\|_{\mathcal{H}^{2m-3}_r} \leq \|\partial^2 E \cdot \partial_v^2 f\|_{\mathcal{H}^{2m-3}_r} \lesssim \|E\|_{H^{2m-1}} \|f\|_{\mathcal{H}^{2m-1}_r}$$

and hence we can take the L^2 norm in time and use (3.12) to obtain that

$$\|\partial_v(\partial^2 E \cdot \nabla_v f)\|_{L^2([0,T],\mathcal{H}^{2m-3}_r)} \leq \Lambda(T,R). \tag{4.29}$$

To estimate the terms in the sum, we use again (3.4), (4.4) and the Sobolev embedding to write

$$\begin{aligned}
&\|\partial^l U \partial^\alpha \partial_v(\partial^2 E \cdot \nabla_v f)\|_{\mathcal{H}^0_r} \\
&\lesssim \|U\|_{L^\infty} \|\partial_v(\partial^2 E \cdot \nabla_v f)\|_{\mathcal{H}^{2m-3}_{2r}} + \|(1+|v|^2)^r \partial^2 E \partial_v^2 f\|_{L^\infty_{x,v}} \|U\|_{\mathcal{H}^{2m-3}_{-r}} \\
&\leq \Lambda(T,R) \|\partial_v(\partial^2 E \cdot \nabla_v f)\|_{\mathcal{H}^{2m-3}_{2r}} + \Lambda(T,R) \|\partial^2 E\|_{L^\infty}.
\end{aligned}$$

Therefore, we get from (3.12) and the Sobolev embedding in x a bound by

$$(1 + \|\partial_v(\partial^2 E \cdot \nabla_v f)\|_{L^2([0,T],\mathcal{H}^{2m-3}_{2r})}) \Lambda(T,R).$$

By using (4.29) (which is still true with r changed into $2r$), we finally obtain that

$$\|J_{2m-3}(\partial_v(\partial^2 E \cdot \nabla_v f))\|_{L^2([0,T],\mathcal{H}^0_r)} \leq \Lambda(T,R). \tag{4.30}$$

It remains to estimate $\mathcal{C}$. By expanding the commutator, we have to estimate terms of the form $\|(1+|v|^2)^{\frac{r}{2}} \partial_x^\beta \nabla_v f \cdot \partial_x^\gamma \partial^2 E\|_{L^2_{x,v}}$ with $|\beta| + |\gamma| \leq 2m-2$, $\beta \neq 0$. If $|\gamma| + 2 + \frac{d}{2} < 2m-1$, by using Sobolev embedding in x and (3.12), we obtain

$$\|(1+|v|^2)^{\frac{r}{2}} \partial_x^\beta \nabla_v f \cdot \partial_x^\gamma \partial^2 E\|_{L^2([0,T],L^2_{x,v})} \leq \Lambda(T,R) \|f\|_{L^\infty([0,T],\mathcal{H}^{2m-1}_r)} \leq \Lambda(T,R).$$

Otherwise, since $|\beta| + 1 + \frac{d}{2} \leq 2 + d < 2m-1$ and $|\gamma| + 2 \leq 2m-1$, we get that

$$\begin{aligned}
&\|(1+|v|^2)^{\frac{r}{2}} \partial_x^\beta \nabla_v f \cdot \partial_x^\gamma \partial^2 E\|_{L^2([0,T],L^2_{x,v})} \\
&\qquad \leq \sup_{[0,T]} \sup_x \Big(\int_{\mathbb{R}^d} |(1+|v|^2)^{\frac{r}{2}} \partial_x^\beta \nabla_v f|^2 \, dv \Big)^{\frac{1}{2}} \|E\|_{L^2([0,T],H^{2m-1}_x)} \\
&\qquad \leq \Lambda(T,R).
\end{aligned}$$

We have thus obtained that

$$\|\mathcal{C}\|_{L^2([0,T],\mathcal{H}^0_r)} \leq \Lambda(T,R).$$

By collecting the last estimate and (4.30), we actually get that

$$\|F_2\|_{L^2([0,T],\mathcal{H}^0_r)} \leq \Lambda(T,R). \tag{4.31}$$

This ends the proof of (4.11).

Estimate of $\mathbf{F_3}$ and $\mathbf{F_4}$. By using similar arguments, we also obtain that

$$\|F_3\|_{L^2([0,T],\mathcal{H}^0_r)} + \|F_4\|_{L^2([0,T],\mathcal{H}^0_r)} \leq \Lambda(T,R). \tag{4.32}$$

□

4.2. Straightening the transport vector field. — We shall now study the equation (4.9) and try to get an estimate of $\int_{\mathbb{R}^d} f_{I,J}\,dv$ which, in view of (4.7), can be used to estimate $\partial_x^{2m}\rho$. The next step consists in using a change of variables in order to straighten the transport vector field and more precisely to come down from the full transport operator $\mathcal{T}$ to a twisted free transport operator of the form

$$\partial_t + \Phi(t,x,v)\cdot\nabla_v.$$

This is the purpose of the following lemma.

Lemma 9. — *Let $f_{I,J}$ be a function satisfying the equation* (4.9)*. Consider $\Phi(t,x,v)$ a smooth solution to the Burgers equation*

$$\partial_t\Phi + \Phi\cdot\nabla_x\Phi = E, \tag{4.33}$$

such that the Jacobian matrix $(\nabla_v\Phi)$ is invertible. Then defining

$$g_{I,J}(t,x,v) := f_{I,J}(t,x,\Phi), \tag{4.34}$$

we obtain that $g_{I,J}$ satisfies the equation

$$\partial_t g_{I,J} + \Phi\cdot\nabla_x g_{I,J} + \partial_x^{\alpha(I,J)}E\cdot(\nabla_v f)(t,x,\Phi) + \mathcal{M}_{I,J}(t,x,\Phi)\mathcal{G} = F_{I,J}(t,x,\Phi), \tag{4.35}$$

where $\mathcal{G} = (\mathcal{G}_{I,J})_{I,J}$, $\alpha(I,J)$ are defined in the statement of Lemma 7 and $\mathcal{M}_{I,J}, F_{I,J}$ are defined in (4.9).

Proof of Lemma 9. — This follows from a direct computation. Using the equations (4.9) and (4.33), we can check that

$$\begin{aligned}\partial_t g_{I,J} + \Phi\cdot\nabla_x g_{I,J} + \partial_x^{\alpha(I,J)}E\cdot(\nabla_v f)(t,x,\Phi) + \mathcal{M}_{I,J}(t,x,\Phi)\mathcal{G}\\ = F_{I,J}(t,x,\Phi) + {}^t(\nabla_v\Phi)^{-1}\nabla_v g_{I,J}\cdot(\partial_t\Phi + \Phi\cdot\nabla_x\Phi - E)\,.\end{aligned}$$

This yields (4.35), because of (4.33). □

We shall now establish Sobolev estimates for the solutions of the Burgers equation (4.33). Choosing the initial condition $\Phi|_{t=0} = v$, we will obtain a control on the deviation from v in Sobolev norms and in particular, observe that $\Phi(t,x,v)$ remains close to v for small enough times.

Lemma 10. — *Assuming that $2m > 3+d$, there exists $\tilde{T}_0 = \tilde{T}_0(R) > 0$ independent of ε such that for every $T < \min(T_0, \tilde{T}_0, T^\varepsilon)$, there is a unique smooth solution on $[0,T]$ of* (4.33) *together with the initial condition $\Phi|_{t=0} = v$.*

Moreover, we have the following uniform estimates for every $T < \min(T_0, \tilde{T}_0, T^\varepsilon)$

$$\sup_{[0,T]}\|\Phi - v\|_{W^{k,\infty}_{x,v}} + \sup_{[0,T]}\left\|\frac{1}{(1+|v|^2)^{\frac12}}\partial_t\Phi\right\|_{W^{k-1,\infty}_{x,v}} \le T^{\frac12}\Lambda(T,R), \quad k < 2m - d/2 - 1. \tag{4.36}$$

Furthermore, we also have that for every $|\alpha| \leq 2m-1$ and $|\beta| \leq 2m-2$,

$$\sup_{[0,T]} \sup_{v} \|\partial^\alpha_{x,v}(\Phi - v)\|_{L^2_x} + \sup_{[0,T]} \sup_{v} \left\| \frac{1}{(1+|v|^2)^{\frac{1}{2}}} \partial^\beta_{x,v} \partial_t \Phi \right\|_{L^2_x} \leq T^{\frac{1}{2}} \Lambda(T,R). \tag{4.37}$$

Proof of Lemma 10. — Let us set $\phi = \Phi - v$. We observe that ϕ solves

$$\partial_t \phi + (v + \phi) \cdot \nabla_x \phi = E$$

with zero initial data. For any $\alpha \in \mathbb{N}^{2d}$, applying ∂^α to the equation, we get that $\partial^\alpha \phi = \partial^\alpha_{x,v} \phi$ satisfies

$$\partial_t \partial^\alpha \phi + v \cdot \nabla_x \partial^\alpha \phi + \phi \cdot \nabla_x \partial^\alpha \phi = \partial^\alpha E - \Big[\sum_{\substack{\beta+\gamma \leq \alpha \\ \gamma \neq \alpha}} c_{\beta,\gamma} \partial^\beta \phi \cdot \nabla_x \partial^\gamma \phi \Big] - [\partial^\alpha, v] \cdot \nabla_x \phi. \tag{4.38}$$

Let us set $M_k(T) = \sup_{[0,T]} \|\phi\|_{W^{k,\infty}_{x,v}}$. Using L^∞ estimates for the transport operator, we obtain from (4.38) that

$$M_k(T) \lesssim T(1 + M_k(T)) M_k(T) + \int_0^T \|E\|_{W^{k,\infty}} \, dt.$$

Since $k + \frac{d}{2} < 2m - 1$, we get by Sobolev embedding in the x variable and (3.12) that

$$M_k(T) \lesssim T(1 + M_k(T)) M_k(T) + T^{\frac{1}{2}} R.$$

Consequently, for $\tilde{T}_0$ sufficiently small depending only on R, we obtain that

$$M_k(T) \leq T^{\frac{1}{2}} \Lambda(T,R).$$

This proves the first part of (4.36). For the estimate on the time derivative, it suffices to use the equation (4.33) and the estimate we have just obtained.

It remains to prove (4.37). We proceed by energy estimates. Using (4.38) for $|\alpha| \leq 2m-1$, multiplying by $\partial^\alpha \phi$ and integrating in x, we obtain from a standard energy estimate (v being only a parameter for the moment)

$$\frac{d}{dt} \|\partial^\alpha \phi\|_{L^2_x} \lesssim (1 + \|\partial_x \phi\|_{L^\infty_x}) \sum_{|\alpha| \leq 2m-1} \|\partial^\alpha \phi\|_{L^2_x} + \|E\|_{H^{2m-1}} + \|\mathcal{C}\|_{L^2_x}$$

where $\mathcal{C}$ is the commutator term

$$\mathcal{C} = \Big[\sum_{\substack{\beta+\gamma \leq \alpha \\ \gamma \neq \alpha}} c_{\beta,\gamma} \partial^\beta \phi \cdot \nabla_x \partial^\gamma \phi \Big].$$

Let us set

$$Q_{2m-1}(T,\phi) = \sup_{[0,T]} \sup_{v} \Big(\sum_{|\alpha| \leq 2m-1} \|\partial^\alpha \phi\|_{L^2_x} \Big).$$

We can then integrate in time and take the sup in time and v to obtain that

$$\begin{aligned} &Q_{2m-1}(T,\phi) \\ &\lesssim Q_{2m-1}(T,\phi)\left(T+\int_0^T \|\partial_x\phi\|_{L^\infty_{x,v}} dt\right) + \int_0^T \|\mathcal{C}\|_{L^\infty_v L^2_x}\, dt + T^{\frac12}\|\rho\|_{L^2([0,T],H^{2m})} \end{aligned}$$

where the last term comes from another use of (3.12). From (4.36), we already have that

$$\|\partial_x\phi\|_{L^\infty_{x,v}} \leq T^{\frac12}\Lambda(T,R),$$

thus it only remains to estimate the commutator term $\mathcal{C}$. For the terms in the sum such that $|\beta| < 2m - \frac{d}{2} - 1$, we can use (4.36) and the fact that $|\gamma| < |\alpha|$ to obtain that

$$\|\partial^\beta\phi\cdot\nabla\partial^\gamma\phi\|_{L^\infty_v L^2_x} \lesssim T^{\frac12}\Lambda(T,R)Q_{2m-1}(T,\phi).$$

In a similar way, when $|\beta| \geq 2m - \frac{d}{2} - 1$, we observe that $1 + |\gamma| \leq \frac{d}{2} < 2m - \frac{d}{2} - 1$ consequently, by using again (4.36), we also obtain that

$$\|\partial^\beta\phi\cdot\nabla\partial^\gamma\phi\|_{L^\infty_v L^2_x} \lesssim T^{\frac12}\Lambda(T,R)Q_{2m-1}(T,\phi).$$

This yields

$$\|\mathcal{C}\|_{L^\infty_v L^2_x} \lesssim T^{\frac12}\Lambda(T,R)Q_{2m-1}(T,\phi)$$

and hence that

$$Q_{2m-1}(T,\phi) \lesssim (T + T^{\frac32}\Lambda(T,R))Q_{2m-1}(T,\phi) + T^{\frac12}R.$$

By taking $\tilde{T}_0$ small enough (depending on R only), we finally obtain that

$$Q_{2m-1}(T,\phi) \lesssim T^{\frac12}\Lambda(T,R)$$

and hence the first part of (4.37) is proven. Again the estimate on the time derivative follows by using the equation (4.33) and the previous estimates.

□

By a change of variable, we can then easily relate the average in v of $f_{I,J}$ to a weighted average of $g_{I,J}$ and therefore we obtain, using (4.7):

Lemma 11. — *We have*

$$\int_{\mathbb{R}^d} g_{I,J}\, J\, dv = \partial_x^{\alpha(I,J)}\rho + \mathcal{R}, \tag{4.39}$$

where $J(t,x,v) := |\det\nabla_v\Phi(t,x,v)|$ *and* $\mathcal{R}$ *still satisfies the estimate* (4.8).

5. Proof of Theorem 1: estimate of $\|\rho\|_{L^2([0,t],H^{2m})}$ by using the Penrose stability condition

Following the reduction of the previous section (from which we keep the same notations), we shall now study the system of equations

$$\partial_t g_{I,J} + \Phi \cdot \nabla_x g_{I,J} + \partial_x^{\alpha(I,J)} E \cdot (\nabla_v f)(t,x,\Phi) + \mathcal{M}_{I,J}(t,x,\Phi)\mathcal{G} = \mathcal{S}_{I,J}, \tag{5.1}$$

with $\mathcal{G} = (g_{I,J})$ and $\mathcal{S}_{I,J}(t,x,v) = F_{I,J}(t,x,\Phi(t,x,v))$. Note that the equations of this system are coupled only through the zero order terms $\mathcal{M}_{I,J}(t,x,\Phi)\mathcal{G}$.

Let us introduce the characteristic flow $X(t,s,x,v)$, $0 \le s,t \le T$

$$\partial_t X(t,s,x,v) = \Phi(t, X(t,s,x,v), v), \quad X(s,s,x,v) = x. \tag{5.2}$$

Note that the velocity variable is only a parameter in this ODE.

We start with estimating the deviation from free transport (that corresponds to the case $\Phi = v$).

Lemma 12. — *For every t, s, $0 \le s \le t \le T$ and T, m, r as in Lemma 10, we can write*

$$X(t,s,x,v) = x + (t-s)\left(v + \tilde{X}(t,s,x,v)\right) \tag{5.3}$$

with $\tilde{X}$ that satisfies the estimate

$$\sup_{t,s\in[0,T]} \left(\|\partial_{x,v}^\alpha \tilde{X}(t,s,x,v)\|_{L^\infty_{x,v}} + \left\| \frac{1}{(1+|v|^2)^{\frac{1}{2}}} \partial_{x,v}^\beta \partial_t \tilde{X}(t,s,x,v) \right\|_{L^\infty_{x,v}} \right) \le T^{\frac{1}{2}} \Lambda(T,R), \tag{5.4}$$

for every $|\alpha| < 2m - d/2 - 1$, $|\beta| < 2m - d/2 - 2$.

Moreover, there exists $\hat{T}_0(R) > 0$ sufficiently small such that for every $T \le \min(T_0, \tilde{T}_0, \hat{T}_0, T^\varepsilon)$, we have that $x \mapsto x + (t-s)\tilde{X}(t,s,x,v)$ is a diffeomorphism and that

$$\sup_{t,s\in[0,T]} \sup_v \left(\|\partial_{x,v}^\alpha \tilde{X}(t,s,x,v)\|_{L^2_x} + \left\| \frac{1}{(1+|v|^2)^{\frac{1}{2}}} \partial_{x,v}^\beta \partial_t \tilde{X}(t,s,x,v) \right\|_{L^2_x} \right) \le T^{\frac{1}{2}} \Lambda(T,R), \tag{5.5}$$

for every $|\alpha| \le 2m-1$, $|\beta| \le 2m-2$. In addition, there exists $\Psi(t,s,x,v)$ such that for t, $s \in [0,T]$ and $T \le \min(T_0, \tilde{T}_0, \hat{T}_0, T^\varepsilon)$, we have

$$X(t,s,x,\Psi(t,s,x,v)) = x + (t-s)v$$

and the following estimates

$$
\begin{aligned}
&\sup_{t,s\in[0,T]}\Big(\|\partial^\alpha_{x,v}(\Psi(t,s,x,v)-v)\|_{L^\infty_{x,v}}+\Big\|\frac{1}{(1+|v|^2)^{\frac12}}\partial^\beta_{x,v}\partial_t\Psi(t,s,x,v)\Big\|_{L^\infty_{x,v}}\Big)\\
&\qquad\qquad\leq T^{\frac12}\Lambda(T,R),\qquad |\alpha|<2m-d/2-1,\ |\beta|<2m-\frac{d}{2}-2\\
&\sup_{t,s\in[0,T]}\sup_v\Big(\|\partial^\alpha_{x,v}(\Psi(t,s,x,v)-v)\|_{L^2_x}+\Big\|\frac{1}{(1+|v|^2)^{\frac12}}\partial^\beta_{x,v}\partial_t\Psi(t,s,x,v)\Big\|_{L^2_x}\Big)\\
&\qquad\qquad\leq T^{\frac12}\Lambda(T,R),\qquad |\alpha|\leq 2m-1,\ |\beta|\leq 2m-2.
\end{aligned}
\tag{5.6}
$$

Proof of Lemma 12. — Let us set $\phi=\Phi-v$ as in the proof of Lemma 10 and $Y(t,s,x,v)=X(t,s,x,v)-x-(t-s)v$. We shall first estimate Y. Since we have

$$
Y(t,s,x,v)=\int_s^t\phi\left(\tau,x+(\tau-s)v+Y(\tau,s,x,v),v\right)\,d\tau,
\tag{5.7}
$$

we deduce from the estimates of Lemma 10 that for $|\alpha|<2m-\frac{d}{2}-1$, we have for $0\leq s,\,t\leq T$,

$$
\begin{aligned}
&\sup_{|\alpha|<2m-\frac{d}{2}-1}\|\partial^\alpha_{x,v}Y(t,s)\|_{L^\infty_{x,v}}\\
&\qquad\leq\left|\int_s^t T^{\frac12}\Lambda(T,R)\big(1+\sup_{|\alpha|<2m-\frac{d}{2}-1}\|\partial^\alpha_{x,v}Y(\tau,s)\|_{L^\infty_{x,v}}\big)\,d\tau\right|.
\end{aligned}
$$

From the Gronwall inequality, this yields

$$
\sup_{|\alpha|<2m-\frac{d}{2}-1}\|\partial^\alpha_{x,v}Y(t,s)\|_{L^\infty_{x,v}}\leq|t-s|T^{\frac12}\Lambda(T,R).
\tag{5.8}
$$

Consequently, we can set $\tilde X(t,s,x,v)=Y(t,s,x,v)/(t-s)$ and deduce from the above estimate that $\tilde X$ verifies the first part of (5.4). To estimate the time derivative, we go back to (5.7). We use a Taylor expansion to write

$$
\begin{aligned}
&\phi\Big(\tau,x+(\tau-s)(v+\tilde X(\tau,s,x,v)),v\Big)\\
&\qquad=\phi(s,x,v)+(\tau-s)\Big(\phi_1(\tau,s,x,v)+\phi_2(\tau,s,x,v)\cdot(v+\tilde X(\tau,s,x,v))\Big)
\end{aligned}
$$

where

$$
\begin{aligned}
\phi_1(\tau,s,x,v)&=\int_0^1\partial_t\phi((1-\sigma)s+\sigma\tau,x,v))d\sigma,\\
\phi_2(\tau,s,x,v)&=\int_0^1 D_x\phi(\tau,x+\sigma(\tau-s)(v+\tilde X(\tau,s,x,v)),v)\,d\sigma.
\end{aligned}
\tag{5.9}
$$

By using (5.7), we thus obtain that

$$
\tilde X(t,s,x,v)=\phi(s,x,v)+\frac{1}{t-s}\int_s^t(\tau-s)Y_1(\tau,s,x,v)\,d\tau
$$

with

$$Y_1(\tau, s, x, v) = \Big(\phi_1(\tau, s, x, v) + \phi_2(\tau, s, x, v) \cdot (v + \tilde{X}(\tau, s, x, v))\Big)\, d\tau \tag{5.10}$$

and thus that

$$\partial_t \tilde{X}(t, s, x, v) = -\frac{1}{(t-s)^2} \int_s^t (\tau - s) Y_1(\tau, s, x, v)\, d\tau + Y_1(t, s, x, v). \tag{5.11}$$

By using (5.8), (4.36) with the same arguments as above, we get that for $|\beta| < 2m - \frac{d}{2} - 2$, the following estimate holds:

$$\left\| \frac{1}{(1+|v|^2)^{\frac12}} \partial_{x,v}^\beta Y_1(\tau, s, x, v) \right\|_{L^\infty_{x,v}} \leq T^{\frac12} \Lambda(T, R).$$

This yields (5.4).

We now turn to the proof of the estimate (5.5). Note that from the estimate (5.8) on Y, we can also ensure that for $\hat{T}_0$ (that depends only on R) sufficiently small, the map $x \mapsto y = x + Y(t, s, x, v)$ is a diffeomorphism with Jacobian $|\det \nabla_x y|$ such that $\frac12 \leq |\det \nabla_x y| \leq 2$.

We shall next prove the estimate (5.5). Let us set

$$M_{2m-1}(t, s) = \sup_v \sup_{|\alpha| \leq 2m-1} \|\partial_{x,v}^\alpha Y(t, s)\|_{L^2_x}.$$

It will be also convenient to introduce the function $g(t, s, x, v) = (x + (t-s)v + Y(t, s, x, v), v)$ so that $\phi(t, x + (t-s)v + Y(t, s, x, v), v) = \phi(t) \circ g(t, s)$. From (5.7), we thus obtain that

$$\begin{aligned}
&\|\partial_{x,v}^\alpha Y(t, s)\|_{L^\infty_v L^2_x} \\
&\quad \lesssim \int_s^t \sum_{k, \beta_1, \cdots, \beta_k} \left\| (D^k_{x,v}\phi(\tau)) \circ g(\tau, s) \cdot \left(\partial_{x,v}^{\beta_1} g(\tau, s), \cdots, \partial_{x,v}^{\beta_k} g(\tau, s)\right) \right\|_{L^\infty_v L^2_x} d\tau
\end{aligned}$$

where the sum is taken on indices such that $k \leq |\alpha| \leq 2m-1$, $\beta_1 + \cdots + \beta_k = |\alpha|$ with for every j, $|\beta_j| \geq 1$ and $|\beta_1| \leq |\beta_2| \leq \cdots \leq |\beta_k|$.

To estimate the right hand side, we first observe that in the sum, if $k \geq 2$, we necessarily have $|\beta_{k-1}| < 2m - \frac{d}{2} - 1$. Indeed, otherwise, there holds $|\beta_1| + \cdots + |\beta_k| \geq 4m - d - 2$ and thus $2m - 1 \geq 4m - d - 2$ which yields $2m \leq d + 1$ and thus is impossible by assumption on m. Next,

– if $k < 2m - \frac{d}{2} - 1$ and $k \geq 2$ we can write thanks to the above observation, Lemma 10 and (5.8) that

$$\begin{aligned}
&\left\| (D^k_{x,v}\phi(\tau)) \circ g(\tau, s) \cdot \left(\partial_{x,v}^{\beta_1} g(\tau, s), \cdots, \partial_{x,v}^{\beta_k} g(\tau, s)\right) \right\|_{L^\infty_v L^2_x} \\
&\leq \|D^k \phi\|_{L^\infty_{x,v}} \|\partial_{x,v}^{\beta_1} g(\tau, s)\|_{L^\infty_{x,v}} \cdots \|\partial_{x,v}^{\beta_{k-1}} g(\tau, s)\|_{L^\infty_{x,v}} \|\partial^{\beta_k} g(\tau, s)\|_{L^\infty_v L^2_x} \\
&\leq T^{\frac12} \Lambda(T, R)(1 + M_{2m-1}(\tau, s)).
\end{aligned}$$

If $k = 1$, the above estimate is obviously still valid.

– if $k > 2m - \frac{d}{2} - 1$, we observe that for every j, $|\beta_j| \leq |\beta_k| \leq 2m - 1 - (k-1) < 1 + \frac{d}{2}$. Thus $|\beta_j| < 2m - \frac{d}{2} - 1$ by the assumption on m and we get by using (5.8) that

$$\|\partial_{x,v}^{\beta_j} g(\tau, s)\|_{L^\infty_{x,v}} \lesssim 1 + T + \|\partial_{x,v}^{\beta_j} Y(\tau, s)\|_{L^\infty_{x,v}} \lesssim \Lambda(T, R).$$

This yields

$$\begin{aligned} \Big\| (D^k_{x,v}\phi(\tau)) \circ g(\tau, s) \cdot \big(\partial_{x,v}^{\beta_1} g(\tau, s), \cdots, \partial_{x,v}^{\beta_k} g(\tau, s)\big) \Big\|_{L^\infty_v L^2_x} \\ \lesssim \big\|(D^k_{x,v}\phi(\tau)) \circ g(\tau, s)\big\|_{L^\infty_v L^2_x} \Lambda(T, R) \\ \lesssim T^{\frac{1}{2}} \Lambda(T, R). \end{aligned}$$

To get the last estimate, we have used that thanks to the choice of $\hat{T}_0$, we can use the change of variable $y = x + Y(t, s, x, v)$ when computing the L^2_x norm of $(D^k_{x,v}\phi(\tau)) \circ g(\tau, s)$ and the estimates of Lemma 10.

By combining the above estimates, we obtain that

$$M_{2m-1}(t, s) \leq (t - s) T^{\frac{1}{2}} \Lambda(T, R) + \int_s^t T^{\frac{1}{2}} \Lambda(T, R) M_{2m-1}(\tau, s)\, d\tau.$$

By using again the Gronwall inequality, we thus obtain that

$$M_{2m-1}(t, s) \leq (t - s) T^{\frac{1}{2}} \Lambda(T, R)$$

and thus by using that $\tilde{X}(t, s, x, v) = Y(t, s, x, v)/(t - s)$, we finally obtain (5.5). To estimate the time derivative, it suffices to combine the above arguments with the expression (5.11) for $\partial_t \tilde{X}$.

To construct Ψ, it suffices to notice that the map $v \mapsto v + \tilde{X}(t, s, x, v)$ is for T sufficiently small a Lipschitz small perturbation of the identity and hence an homeomorphism on $\mathbb{R}^d$. We can define Ψ as its inverse. The claimed regularity follows easily by using the same composition estimates as above and the regularity of $\tilde{X}$. □

Define now the tensor $\mathcal{M}$ by the formula $(\mathcal{M}H)_{I,J} = \mathcal{M}_{I,J} H$ for all $I, J \in \{1, \cdots, d\}^m$ (with $\mathcal{M}_{I,J}$ defined in (4.10)) and for $0 \leq s, t \leq T$, $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$, introduce the operator $\mathfrak{M}(t, s, x, v)$ as the solution of

$$\partial_t \mathfrak{M}(t, s, x, v) = -\mathcal{M}(t, x, \Phi(t, x, v)) \mathfrak{M}(t, s, x, v), \quad \mathfrak{M}(s, s, x, v) = I. \tag{5.12}$$

Note that by a straightforward Gronwall type argument and (4.4)-(4.36), we have

$$\sup_{0 \leq s, t \leq T} \Big(\|\mathfrak{M}\|_{W^{k,\infty}_{x,v}} + \|\partial_t \mathfrak{M}\|_{W^{k,\infty}_{x,v}} + \|\partial_s \mathfrak{M}\|_{W^{k,\infty}_{x,v}} \Big) \leq \Lambda(T, R), \qquad k < 2m - d/2 - 2. \tag{5.13}$$

We shall now show that the study of (5.1) can be reduced to that of a system of integral equations with a well controlled remainder.

Lemma 13. — *For a smooth vector field $G(t,s,x,v)$, define the following integral operators K_G acting on functions $F(t,x)$:*

$$K_G(F)(t,x) = \int_0^t \int (\nabla_x F)(s, x-(t-s)v) \cdot G(t,s,x,v)\, dvds.$$

For f solving (1.1) *and $\rho = \int f\, dv$, the function $\partial_x^{\alpha(I,J)}\rho$ satisfies an equation under the form*

$$\partial_x^{\alpha(I,J)}\rho = \sum_{K,L\in\{1,\cdots,d\}^m} K_{H_{(K,L),\,(I,J)}}((I-\varepsilon^2\Delta)^{-1}\partial_x^{\alpha(K,L)}\rho) + \mathcal{R}_{I,J}, \tag{5.14}$$

with

$$\begin{aligned} &H_{(K,L),\,(I,J)} \\ &:= \mathfrak{M}_{(K,L),\,(I,J)}(t,s,x-(t-s)v, \Psi(s,t,x,v))(\nabla_v f)(s, x-(t-s)v, \Psi(s,t,x,v)) \\ &\qquad \times |\det \nabla_v \Phi(t,x,\Psi(s,t,x,v))||\det \nabla_v \Psi(s,t,x,v)|, \end{aligned} \tag{5.15}$$

and the remainder $\mathcal{R}_{I,J}$ satisfies for $T<\min(T_0,\tilde{T}_0,\hat{T}_0,T^\varepsilon)$ the estimate

$$\|\mathcal{R}_{I,J}\|_{L^2([0,T],L^2_x)} \lesssim T^{\frac12}\Lambda(T,R). \tag{5.16}$$

Proof of Lemma 13. — Let us introduce for notational brevity

$$\eta(t,x,v) := \left(\partial_x^{\alpha(I,J)}E(t,x)\cdot\nabla_v f(s,x,\Phi(t,x,v))\right)_{I,J}.$$

Using the classical characteristics method, we get that $\mathcal{G}$ satisfying (5.1) solves the integral equation

$$\begin{aligned} \mathcal{G}(t,x,v) = \mathfrak{M}(t,0,x,v)\mathcal{G}^0(X(0,t,x,v),v) &- \int_0^t \mathfrak{M}(t,s,x,v)\mathcal{S}(s,X(s,t,x,v),v)\, ds \\ &- \int_0^t \mathfrak{M}(t,s,x,v)\eta(s,X(s,t,x,v),v)ds \end{aligned}$$

with $\mathcal{G}^0 = (g^0_{I,J})$ and $\mathcal{S} = (\mathcal{S}_{I,J})$. Hence, after multiplying by $J(t,x,v) = |\det\nabla_v\Phi(t,x,v)|$ and integrating in v, we get that

$$\int_{\mathbb{R}^d} \mathcal{G}(t,x,v)J(t,x,v)dv = \mathcal{I}_0 + \mathcal{I}_F - \int_0^t \mathfrak{M}(t,s,x,v)\eta(s,X(s,t,x,v),v)J(t,x,v)\, dvds \tag{5.17}$$

with

$$\begin{aligned} \mathcal{I}_0 &:= \int_{\mathbb{R}^d} \mathfrak{M}(t,0,x,v)\mathcal{G}^0(X(0,t,x,v),v)J(t,x,v)\, dv, \\ \mathcal{I}_F &:= -\int_0^t\int_{\mathbb{R}^d} \mathfrak{M}(t,s,x,v)\mathcal{S}(s,X(s,t,x,v),v)J(t,x,v)\, dvds. \end{aligned}$$

We shall estimate $\mathcal{I}_0$ and $\mathcal{I}_F$. First by using the estimates (5.13) and (4.36), it follows that for all $x \in \mathbb{T}^d$,

$$\begin{aligned}&\left|\int \mathfrak{M}(t,0,x,v)\mathcal{G}^0(X(0,t,x,v),v)J(t,x,v)\,dv\right|\\ &\leq \Lambda(T,R)\sum_{I,J}\int |g^0_{I,J}(X(0,t,x,v),v)|\,dv.\end{aligned}$$

Therefore, we obtain that

$$\|\mathcal{I}_0\|_{L^2([0,T],L^2_x)} \leq \Lambda(T,R)\sum_{I,J}\left\|\int_v \|g^0_{I,J}(X(0,t,\cdot,v),v)\|_{L^2_x}\,dv\right\|_{L^2(0,T)}.$$

By using the change of variable in x, $y = X(0,t,x,v)+tv = x - t\tilde{X}(0,t,x,v)$ and Lemma 12, we obtain that

$$\|g^0_{I,J}(X(0,t,\cdot,v),v)\|_{L^2_x} \leq \Lambda(T,R)\|g^0_{I,J}(\cdot - tv,v)\|_{L^2} \leq \Lambda(T,R)\|g^0_{I,J}(\cdot,v)\|_{L^2_x}$$

and hence, we get from Cauchy-Schwarz that

$$\|\mathcal{I}_0\|_{L^2([0,T],L^2_x)} \leq T^{\frac{1}{2}}\Lambda(T,R)\left(\int_{\mathbb{R}^d}\frac{dv}{(1+|v|^2)^r}\right)^{\frac{1}{2}}\sum_{I,J}\|g^0_{I,J}\|_{\mathcal{H}^0_r}.$$

By using again Lemma 10 and the fact that at $t=0$ we have that $L^{(I,J)} = \partial_x^{\alpha(I,J)}$ we get that $\|g^0_{I,J}\|_{\mathcal{H}^0_r} \leq \Lambda(T,R)\|f^0_{I,J}\|_{\mathcal{H}^0_r} \leq \Lambda(T,R)\|f^0\|_{\mathcal{H}^{2m}_r}$ and hence we finally obtain that

$$\|\mathcal{I}_0\|_{L^2([0,T],L^2_x)} \leq T^{\frac{1}{2}}\Lambda(T,R).$$

By using similar arguments, we can estimate $\mathcal{I}_F$. Indeed, we can use successively (5.13), the change of variable $x \mapsto X(s,t,x,v)$ with Lemma 12 and Cauchy-Schwarz to obtain that

$$\begin{aligned}\|\mathcal{I}_F\|_{L^2([0,T],L^2_x)} &\leq \Lambda(T,R)\sum_{I,J}\left\|\int_0^t\int_{\mathbb{R}^d}\|\mathcal{S}_{I,J}(s,X(s,t,\cdot,v),v)\|_{L^2_x}\,dvds\right\|_{L^2(0,T)}\\ &\leq \Lambda(T,R)\sum_{I,J}\left\|\int_0^t\int_{\mathbb{R}^d}\|\mathcal{S}_{I,J}(s,\cdot,v)\|_{L^2_x}\,dvds\right\|_{L^2(0,T)}\\ &\leq \Lambda(T,R)\left\|\int_0^t\|\mathcal{S}(s)\|_{\mathcal{H}^0_r}\,ds\right\|_{L^2(0,T)}\\ &\leq \Lambda(T,R)\,T\,\|\mathcal{S}\|_{L^2([0,T],\mathcal{H}^0_r)}.\end{aligned}$$

Finally, since $\mathcal{S}_{I,J}(t,x,v) = F_{I,J}(t,x,\Phi(t,x,v))$, we can use Lemma 10 and (4.11) to obtain that

$$\|\mathcal{S}\|_{L^2([0,T],\mathcal{H}^0_r)} \leq \Lambda(T,R)\|F\|_{L^2([0,T],\mathcal{H}^0_r)} \leq \Lambda(T,R).$$

We have thus proven that

$$\|\mathcal{I}_F\|_{L^2([0,T],L^2_x)} \leq T\Lambda(T,R).$$

By using Lemma 11, we eventually obtain from (5.17) and the above estimates that

$$\partial_x^{\alpha(I,J)}\rho = \mathcal{R}_{I,J} - \int_0^t \int \sum_{K,L} \mathfrak{M}_{(K,L),\,(I,J)}(t,s,x,v)(\partial_x^{\alpha(K,L)}E)(s,X(s,t,x,v)) \cdot (\nabla_v f)\left((s,X(s,t,x,v),\Phi(s,X(s,t,x,v),v)\right) J(t,x,v)\,dvds$$

with

$$\|\mathcal{R}_{I,J}\|_{L^2([0,T],L^2_x)} \lesssim T^{\frac{1}{2}}\Lambda(T,R). \tag{5.18}$$

Thanks to Lemma 12, we can use the change of variable $v = \Psi(s,t,x,w)$ (and relabel w by v) to end up with the integral equation

$$\partial_x^{\alpha(I,J)}\rho = -\int_0^t \int \sum_{K,L} (\partial_x^{\alpha(K,L)}E)(s,x-(t-s)v)\cdot H_{(K,L),\,(I,J)}(t,s,x,v)\,dvds + \mathcal{R}_0 \tag{5.19}$$

with

$$H_{(K,L),\,(I,J)} = \mathfrak{M}_{(K,L),\,(I,J)}(t,s,x,\Psi(s,t,x,v))(\nabla_v f)(s,x-(t-s)v,\Psi(s,t,x,v)) J(t,x,\Psi(s,t,x,v))\tilde{J}(s,t,x,v), \tag{5.20}$$

and $\tilde{J}(s,t,x,v) = |\det \nabla_v \Psi(s,t,x,v)|$, which, recalling the definition of the electric field $E = -\nabla(I-\varepsilon^2\Delta)^{-1}(\rho-1)$, corresponds to the claimed formula (5.14).

□

We shall now study the boundedness of the operators K_G for functions in $L^2([0,T],L^2_x)$. Although K_G seems to feature a loss of one derivative in x, we shall see that if the function G is sufficiently smooth, then K_G is actually a bounded operator on $L^2([0,T],L^2_x)$. This means that we can recover the apparent loss of derivative by using the averaging in v, which is reminiscent of averaging lemmas (note that we nevertheless require regularity on G).

Proposition 2. — *There exists $C>0$ such that for every $T>0$ and every G with*

$$\|G\|_{T,s_1,s_2} := \sup_{0\le t\le T}\left(\sum_k \sup_{0\le s\le T}\sup_\xi \left((1+|k|)^{s_2}(1+|\xi|)^{s_1}|(\mathcal{F}_{x,v}G)(t,s,k,\xi)|\right)^2\right)^{\frac{1}{2}} < +\infty, \tag{5.21}$$

for $s_1>1$, $s_2>d/2$, we have the estimate

$$\|K_G(F)\|_{L^2([0,T],L^2_x)} \le C\|G\|_{T,s_1,s_2}\|F\|_{L^2([0,T],L^2_x)}, \quad \forall F\in L^2([0,T],L^2_x).$$

Remark 3. — *For practical use, it is convenient to relate* $\|G\|_{T,s_1,s_2}$ *to a more tractable norm. A first way to do it is to observe that if* $p > 1 + d$, $\sigma > d/2$, *we can find* $s_2 > d/2$, $s_1 > 1$ *such that*

$$\Big((1+|k|)^{s_2}(1+|\xi|)^{s_1}|(\mathcal{F}_{x,v}G)(t,s,k,\xi)|\Big)^2 \leq \frac{1}{(1+|k|)^{2s_2}}\|G(t,s)\|^2_{\mathcal{H}^p_\sigma}$$

and thus we obtain that

$$\|G\|_{T,s_1,s_2} \lesssim \sup_{0\leq s,\, t\leq T}\|G(t,s)\|_{\mathcal{H}^p_\sigma}.$$

Note that this requires roughly $1+d$ *derivatives of the function* G. *In the following, we shall need only the above Proposition in the following two cases for which we can reduce the number of derivatives needed on the function* G.

– *If* $G(t,s,x,v) = G(t,x,v)$ *is independent of* s, *then we have thanks to the Bessel-Parseval identity that*

$$\|G\|_{T,s_1,s_2} \leq \sup_{0\leq t\leq T}\|G(t)\|_{\mathcal{H}^p_\sigma} \tag{5.22}$$

for any integer p *such that* $p > 1 + \frac{d}{2}$ *and any* σ, $\sigma > \frac{d}{2}$.

– *If* $G(t,t,x,v) = 0$, *since*

$$\begin{aligned}&\left((1+|k|)^{s_2})(1+|\xi|)^{s_1}\,|\mathcal{F}_{x,v}(G)(t,s,k,\xi)|\right)^2\\ &\leq T\left|\int_s^t \left((1+|k|)^{s_2}(1+|\xi|)^{s_1}\,|\mathcal{F}_{x,v}(\partial_s G)(t,\tau,k,\xi)|\right)^2 d\tau\right|\\ &\leq T\left|\int_0^t \left((1+|k|)^{s_2}(1+|\xi|)^{s_1}\,|\mathcal{F}_{x,v}(\partial_s G)(t,\tau,k,\xi)|\right)^2 d\tau\right|,\end{aligned}$$

we obtain, by using again the Bessel-Parseval identity that

$$\|G\|_{T,s_1,s_2} \leq T^{\frac12}\sup_{0\leq t\leq T}\left(\int_0^t \|\partial_s G(t,s)\|^2_{\mathcal{H}^p_\sigma}\,ds\right)^{\frac12}$$

and hence that

$$\|G\|_{T,s_1,s_2} \leq T\sup_{0\leq t\leq T}\sup_{0\leq s\leq t}\|\partial_s G(t,s)\|_{\mathcal{H}^p_\sigma} \tag{5.23}$$

with $p > 1 + \frac{d}{2}$ *and* $\sigma > \frac{d}{2}$.

Proof of Proposition 2. — By using Fourier series in x, we can write that

$$F(t,x) = \sum_{k\in\mathbb{Z}}\hat{F}_k(t)e^{ik\cdot x}.$$

This yields

$$K_G F(t,x) = \int_0^t \sum_k \hat{F}_k(s) e^{ik\cdot x} ik \cdot \int e^{-ik\cdot v(t-s)} G(t,s,x,v)\, dv\, ds \tag{5.24}$$

$$= (2\pi)^d \int_0^t \sum_k \hat{F}_k(s) e^{ik\cdot x} ik \cdot (\mathcal{F}_v G)(t,s,x,k(t-s))\, ds \tag{5.25}$$

where $\mathcal{F}_v G$ stands for the Fourier transform of $G(t,s,x,v)$ with respect to the last variable. Next, expanding also G in Fourier series in the x variable, we get that

$$K_G F(t,x) = (2\pi)^d \sum_k e^{ik\cdot x} \sum_l e^{il\cdot x} \int_0^t \hat{F}_k(s) ik \cdot (\mathcal{F}_{x,v} G)(t,s,l,k(t-s))\, ds.$$

Changing l into $l+k$ in the second sum, we can also write this expression as

$$K_G F(t,x) = (2\pi)^d \sum_l e^{il\cdot x} \Big(\sum_k \int_0^t \hat{F}_k(s) ik \cdot (\mathcal{F}_{x,v} G)(t,s,l-k,k(t-s))\, ds \Big).$$

From the Bessel-Parseval identity, this yields

$$\|K_G\|^2_{L^2_x} = (2\pi)^d \sum_l \Big| \sum_k \int_0^t \hat{F}_k(s) ik \cdot (\mathcal{F}_{x,v} G)(t,s,l-k,k(t-s))\, ds \Big|^2.$$

By using Cauchy-Schwarz in time and k, we next obtain that

$$\|K_G\|^2_{L^2_x} \lesssim \sum_l \Big(\sum_k \int_0^t |\hat{F}_k(s)|^2 \, |k \cdot (\mathcal{F}_{x,v} G)(t,s,l-k,k(t-s))|\, ds \\ \cdot \sum_k \int_0^t |k \cdot (\mathcal{F}_{x,v} G)(t,s,l-k,k(t-s))|\, ds \Big).$$

By integrating in time, this yields

(5.26)

$$\|K_G\|^2_{L^2([0,T],L^2_x)} \lesssim \sum_l \int_0^T \int_0^t \sum_k |\hat{F}_k(s)|^2 \, |k \cdot (\mathcal{F}_{x,v} G)(t,s,l-k,k(t-s))|\, ds\, dt \\ \cdot \sup_l \sup_{t\in[0,T]} \int_0^t \sum_k |k \cdot (\mathcal{F}_{x,v} G)(t,s,l-k,k(t-s))|\, ds. \leq I \cdot II.$$

For the second term in the above product that is II, we observe that

$$\sup_l \sup_{t\in[0,T]} \int_0^t \sum_k |k \cdot (\mathcal{F}_{x,v} G)(t,s,l-k,k(t-s))|\, ds \leq$$

$$\sup_l \sup_{t\in[0,T]} \sum_k \Big(\sup_{0\leq s\leq t} \sup_\xi (1+|\xi|)^{s_1} |(\mathcal{F}_{x,v} G)(t,s,l-k,\xi)| \int_0^t \frac{|k|}{(1+|k|(t-s))^{s_1}}\, ds \Big),$$

and by choosing $s_1 > 1$, since

$$\int_0^t \frac{|k|}{(1+|k|(t-s)|)^{s_1}}\, ds \leq \int_0^{+\infty} \frac{1}{(1+\tau^{s_1})}\, d\tau < +\infty$$

is independent of k, we obtain

$$\begin{aligned}\sup_l \sup_{t\in[0,T]} \int_0^t \sum_k |k\cdot(\mathcal{F}_{x,v}G)(t,s,l-k,k(t-s))|\, ds \\ \leq \sup_l \sup_{t\in[0,T]} \sum_k \sup_{0\leq s\leq t} \sup_\xi (1+|\xi|)^{s_1} |(\mathcal{F}_{x,v}G)(t,s,l-k,\xi)|.\end{aligned}$$

Next, by choosing $s_2 > d/2$ and by using Cauchy-Schwarz, this finally yields

$$II \leq \sup_{t\in[0,T]} \Big(\sum_k \sup_{0\leq s\leq t} \sup_\xi \Big((1+|k|)^{s_2}(1+|\xi|)^{s_1}|(\mathcal{F}_{x,v}G)(t,s,k,\xi)|\Big)^2\Big)^{\frac12}. \tag{5.27}$$

It remains to estimate I in the right-hand side of (5.26). By using Fubini, we have

$$\begin{aligned}&\sum_l \int_0^T \int_0^t \sum_k |\hat{F}_k(s)|^2\, |k\cdot(\mathcal{F}_{x,v}G)(t,s,l-k,k(t-s))|\, ds\, dt \\ &= \int_0^T \sum_k |\hat{F}_k(s)|^2 \int_s^T \sum_l |k||(\mathcal{F}_{x,v}G)(t,s,l-k,k(t-s))|\, dt\, ds \\ &\leq \|F\|^2_{L^2([0,T],L^2_x)} \sup_k \sup_{0\leq s\leq t} \int_s^T \sum_l |k||(\mathcal{F}_{x,v}G)(t,s,l-k,k(t-s))|\, dt.\end{aligned}$$

Next, by choosing $s_1 > 1$ and $s_2 > d/2$ as above, we observe that

$$\begin{aligned}&\sup_k \sup_{0\leq s\leq T} \int_s^T \sum_l |k||(\mathcal{F}_{x,v}G)(t,s,l-k,k(t-s))|\, dt \\ &\leq \sup_k \sup_{0\leq s\leq T} \int_s^T \frac{|k|}{(1+|k|(t-s))^{s_1}} \sum_l \sup_\xi (1+|\xi|)^{s_1}|(\mathcal{F}_{x,v}G)(t,s,l-k,\xi)|\, dt \\ &\leq \sup_k \sup_{0\leq s\leq T} \int_s^T \frac{|k|\, dt}{(1+|k|(t-s))^{s_1}} \\ &\qquad \times \sup_{0\leq s\leq T} \sup_{s\leq t\leq T} \left(\sum_m \sup_\xi \Big((1+|m|)^{s_2}(1+|\xi|)^{s_1}|(\mathcal{F}_{x,v}G)(t,s,m,\xi)|\Big)^2\right)^{\frac12}.\end{aligned}$$

Since we have again

$$\sup_k \sup_{0\leq s\leq T} \int_s^T \frac{|k|\, dt}{(1+|k|(t-s))^{s_1}} \leq \int_0^{+\infty} \frac{d\tau}{(1+\tau)^{s_1}}\, d\tau < +\infty,$$

we have proven that

(5.28)
$$\begin{aligned} I &\lesssim \|F\|^2_{L^2([0,T],L^2_x)} \\ &\quad \times \sup_{0\le s\le T}\sup_{s\le t\le T}\left(\sum_m \sup_\xi \Big((1+|m|)^{s_2}(1+|\xi|)^{s_1}|(\mathcal{F}_{x,v}G)(t,s,m,\xi)|\Big)^2\right)^{\frac12} \\ &\lesssim \|F\|^2_{L^2([0,T],L^2_x)} \\ &\quad \times \sup_{0\le t\le T}\left(\sum_m \sup_{0\le s\le t}\sup_\xi \Big((1+|m|)^{s_2}(1+|\xi|)^{s_1}|(\mathcal{F}_{x,v}G)(t,s,m,\xi)|\Big)^2\right)^{\frac12}. \end{aligned}$$

We finally get the result by combining (5.26), (5.28) and (5.27).

□

We can then use Proposition 2 to simplify the system of equations (5.14) for ρ in Lemma 13.

Lemma 14. — *Assume $2m > m_0$ and $2r > r_0$. For f solving (1.1) and $\rho = \int f\, dv$, for every $I, J \in \{1, \cdots, d\}^m$, the function $\partial_x^{\alpha(I,J)}\rho$ satisfies an equation under the form*

$$\partial_x^{\alpha(I,J)}\rho = K_{\nabla_v f^0}((I-\varepsilon^2\Delta)^{-1}\partial_x^{\alpha(I,J)}\rho) + R_{I,J}, \tag{5.29}$$

where the remainder $R_{I,J}$ satisfies

$$\|R_{I,J}\|_{L^2([0,T],L^2_x)} \lesssim T^{\frac12}\Lambda(T,R). \tag{5.30}$$

for $T < \min(T_0, \tilde{T}_0, \hat{T}_0, T^\varepsilon)$.

Proof of Lemma 14. — We keep the same notations as in Lemma 13 and in particular we use the expression (5.20). We can first write that

$$H_{(I,J),(K,L)}(t,s,x,v) = H_{(I,J),(K,L)}(t,t,x,v) + H^1_{(I,J),(K,L)}(t,s,x,v) \tag{5.31}$$

with

$$H^1_{(I,J),(K,L)}(t,s,x,v) = H_{(I,J),(K,L)}(t,s,x,v) - H_{(I,J),(K,L)}(t,t,x,v).$$

Since $H^1_{(I,J),(K,L)}(t,t,x,v) = 0$, we can use (5.23) in Remark 3 to get that

$$\begin{aligned} &\|K_{H^1_{(I,J),(K,L)}}(I-\varepsilon^2\Delta)^{-1}\partial_x^{\alpha(K,L)}\rho\|_{L^2([0,T],L^2_x)} \\ &\qquad\qquad \lesssim T\sup_{t,s}\|\partial_s H^1_{(I,J),(K,L)}(t,s)\|_{H^p_\sigma}\|\rho\|_{L^2([0,T],H^{2m})} \end{aligned}$$

with $p = 1 + p_0$ (p_0 being defined in (1.9)) and σ such that $\sigma > \frac{d}{2}$ and $1 + \sigma < 2r$. We thus have to estimate $\sup_{t,s}\|\partial_s H^1_{(I,J),(K,L)}(t,s)\|_{\mathcal{H}^p_\sigma}$.

Note that, by assumption on m, we have that $p + 2 < 2m - \frac{d}{2}$. We use again the notation

$$J(t,x,v) = |\det(\nabla_v\Phi(t,x,v))|, \quad \tilde{J}(s,t,x,v) = |\det(\nabla_v\Psi(s,t,x,v))|.$$

According to (5.13), (5.6), (4.36), we can always put the terms involving $\mathfrak{M}$, Ψ, J, $\tilde{J}$ and their derivatives in L^∞, except when all the derivatives hit J or $\tilde{J}$. Note that due to the expression (5.20), to compute $\partial_s H^1$, we need the derivative of Ψ and $\mathfrak{M}$ with respect to their first argument so that we actually need estimates of $\partial_t \Psi$ in view of our previous notation. This yields

$$\begin{aligned}
&\|\partial_s H^1_{(I,J),\,(K,L)}(t,s)\|_{\mathcal{H}^p_\sigma} \\
&\qquad \leq \Lambda(T,R)\Big(\|\partial_t \nabla_v f\|_{\mathcal{H}^p_\sigma} + \Big\|(1+|v|^2)^{\frac{1}{2}}\nabla_v f\Big\|_{\mathcal{H}^{p+1}_\sigma} \\
&\qquad + \sum_{|\alpha|=p} \big\||\nabla_v f(s,x-(t-s)v,\Psi)|\,|(\partial^\alpha_{x,v}\nabla_v J)(t,\cdot,\Psi)|\big\|_{\mathcal{H}^0_\sigma} \\
&\qquad + \Big\||\nabla_v f(s,x-(t-s)v,\Psi)|\,|\partial^\alpha_{x,v}\partial_s \tilde{J}|\Big\|_{\mathcal{H}^0_\sigma}\Big).
\end{aligned} \tag{5.32}$$

Note that to obtain this estimate, we have used that integrals under the form

$$I = \left|\int_{\mathbb{T}^d\times\mathbb{R}^d} |g(x-v(t-s),\Psi(s,t,x,v))|^2(1+|v|^2)^n\,dxdv\right|$$

with $n=\sigma$ or $\sigma+1$ can be bounded by $\Lambda(T,R)\|g\|^2_{\mathcal{H}^0_n}$. Indeed, by setting, $v\mapsto w = \Psi(s,t,x,v)$ and by using Lemma 12 (in particular the fact that the Jacobian of the change of variable is bounded and the fact that $|w-v|$ is bounded), we get

$$I \leq \Lambda(T,R)\int_{\mathbb{T}^d\times\mathbb{R}^d} |g(X(s,t,x,w),w)|^2(1+|w|^2)^n\,dxdw.$$

Next, we can use again Lemma 12 and the change of variable $x\mapsto y = X(s,t,x,w)$ to finally obtain

$$I \leq \Lambda(T,R)\|g\|^2_{\mathcal{H}^0_n}.$$

Going back to (5.32), we observe that by using the equation (1.1), we get that

$$\|\partial_t\nabla_v f\|_{\mathcal{H}^p_\sigma} + \Big\|(1+|v|^2)^{\frac{1}{2}}\nabla_v f\Big\|_{\mathcal{H}^{p+1}_\sigma} \lesssim \|f\|_{\mathcal{H}^{2m-1}_{2r}}$$

since $2m > m_0$ implies that $2m \geq 4+p_0$. Also, by using again L^∞ estimates, we have for $|\alpha| = p$,

$$\begin{aligned}
&\Big\||\nabla_v f(s,x-(t-s)v,\Psi)|\,|(\partial^\alpha_{x,v}\nabla_v J)(t,\cdot,\Psi)|\Big\|_{\mathcal{H}^0_\sigma} \\
&\qquad + \Big\||\nabla_v f(s,x-(t-s)v,\Psi)|\,|\partial^\alpha_{x,v}\partial_s\tilde{J}|\Big\|_{\mathcal{H}^0_\sigma} \\
&\qquad \leq \Lambda(T,R)\Big(\big\||\nabla_v f(s,x-(t-s)v,\Psi)|\,|(\partial^\alpha_{x,v}\nabla^2_v\Phi)(t,\cdot,\Psi)|\big\|_{\mathcal{H}^0_\sigma} \\
&\qquad + \big\||\nabla_v f(s,x-(t-s)v,\Psi)|\,|\partial^\alpha_{x,v}\partial_s\nabla_v\Psi|\big\|_{\mathcal{H}^0_\sigma}\Big)
\end{aligned}$$

and we estimate the above right-hand side by

$$\Lambda(T,R)\|(1+|v|^2)^{\frac{\sigma+1}{2}}\nabla_v f\|_{L^\infty_{x,v}} \times \Big(\|(\partial^\alpha_{x,v}\nabla_v^2\Phi)(t,\cdot,\Psi)\|_{L^\infty_v L^2_x} + \|(1+|v|^2)^{-\frac{1}{2}}\partial^\alpha_{x,v}\partial_s\nabla_v\Psi\|_{L^\infty_v L^2_x}\Big).$$

Since $2m \geq 4+p_0$, the above expression can be again finally bounded by $\Lambda(T,R)$ by using (5.6), (4.37) and the Sobolev embedding in x, v to estimate $\|(1+|v|^2)^{\frac{\sigma+1}{2}}\nabla_v f\|_{L^\infty_{x,v}}$. We have thus proven that

$$\|K_{H^1_{(I,J),\,(K,L)}}(I-\varepsilon^2\Delta)^{-1}\partial_x^{\alpha(K,L)}\rho\|_{L^2([0,T],L^2_x)} \lesssim T\Lambda(T,R)$$

and as a consequence, that this term can be included in the remainder.

In view of (5.31) and the above estimate, since

$$H_{(I,J),\,(K,L)}(t,t,x,v) = \delta_{(I,J),\,(K,L)}\nabla_v f(t,x,v)J(t,x,v),$$

where δ denotes the Kronecker function, it follows that the integral system (5.14) reduces to

$$\partial_x^{\alpha(I,J)}\rho = K_{\nabla_v f(t)J(t)}((I-\varepsilon^2\Delta)^{-1}\partial_x^{\alpha(I,J)})\rho + \mathcal{R}^1_{(I,J)}$$

with $\mathcal{R}^1_{(I,J)}$ that satisfies

$$\left\|\mathcal{R}^1_{(I,J)}\right\|_{L^2([0,T],L^2_x)} \leq T^{\frac{1}{2}}\Lambda(T,R).$$

We can further simplify this integral equation by writing

$$\begin{aligned}&\nabla_v f(t,x,v)J(t,x,v)\\ &= \nabla_v f^0(x,v) + \big(\nabla_v f(t,x,v)-\nabla_v f^0(x,v)\big)\,J(t,x,v) + \nabla_v f^0(x,v)(J(t,x,v)-1).\end{aligned}$$

Let us set

$$G(t,x,v) = \big(\nabla_v f(t,x,v)-\nabla_v f^0(x,v)\big)\,J(t,x,v) + \nabla_v f^0(x,v)(J(t,x,v)-1).$$

By using Proposition 2 and (5.22) in Remark 3, we obtain that

$$\|K_G((I-\varepsilon^2\Delta)^{-1}\partial_x^{\alpha(I,J)}\rho)\|_{L^2([0,T],L^2_x)} \leq (\sup_{[0,T]}\|G(t)\|_{\mathcal{H}^p_\sigma})\|\rho\|_{L^2([0,T],H^{2m})},$$

with $p = 1+p_0$ and $\sigma > \frac{d}{2}$, $1+\sigma \leq 2r$. From the definition of G, we find

$$\|G(t)\|_{\mathcal{H}^p_\sigma} \lesssim \|\nabla_v f(t,\cdot)-\nabla_v f^0\|_{\mathcal{H}^p_\sigma}\|J(t)\|_{W^{p,\infty}} + \|\nabla_v f^0\|_{\mathcal{H}^p_\sigma}\|J(t)-1\|_{W^{p,\infty}}.$$

Using the Vlasov equation in (1.1), we write

$$f(t)-f^0 = \int_0^t \partial_t f(s)\,ds = -\int_0^t (v\cdot\nabla_x f(s) + E\cdot\nabla_v f(s))\,ds. \tag{5.33}$$

Since we have $1+p < 2m-\frac{d}{2}-1$, $2+p \leq 2m-1$, $1+\sigma \leq 2r$, we obtain by using (4.36) and (5.33) that

$$\sup_{[0,T]}\|G(t)\|_{\mathcal{H}^p_\sigma} \leq \Lambda(T,R)\Big(T\sup_{[0,T]}\|\partial_t f\|_{\mathcal{H}^{p+1}_\sigma} + T^{\frac{1}{2}}\Big) \leq T^{\frac{1}{2}}\Lambda(T,R).$$

Consequently, we obtain

$$\|K_G(\partial_x^{\alpha(I,J)}\rho)\|_{L^2([0,T],L^2_x)} \leq T^{\frac{1}{2}}\Lambda(T,R),$$

and this term can be put in the remainder. Gathering all pieces together, we obtain (5.29) with (5.30). This ends the proof.

□

We therefore proceed with the study of the integral scalar equation

$$\tilde{h}(t,x) = K_{\nabla_v f^0}((I-\varepsilon^2\Delta)^{-1}\tilde{h}) + \tilde{R}(t,x), \quad 0 \leq t \leq T, \tag{5.34}$$

where $\tilde{R}$ is a given source term. It will be useful to introduce a positive parameter γ (which will be chosen large enough but independent of ε) and to set

$$\tilde{h}(t,x) = e^{\gamma t}h(t,x), \quad \tilde{R}(t,x) = e^{\gamma t}\mathcal{R}(t,x) \tag{5.35}$$

so that (5.34) becomes

$$h(t,x) = e^{-\gamma t}K_{\nabla_v f^0}(e^{\gamma t}(I-\varepsilon^2\Delta)^{-1}h) + \mathcal{R}(t,x), \quad 0 \leq t \leq T \tag{5.36}$$

Without loss of generality, we can assume that $\mathcal{R}$ is equal to zero for $t < 0$ and for $t > T$ and we shall also set $h = 0$ for all $t < 0$. Note that this does not affect the value of h on $[0,T]$. This allows us to study the equation for $t \in \mathbb{R}$. Our aim is to prove that if the Penrose condition is satisfied by f^0 then we can estimate h in $L^2_{t,x}$ with respect to R in $L^2_{t,x}$.

One first key step is to relate $e^{-\gamma t}K_{\nabla_v f^0}(e^{\gamma t}\cdot)$ to a pseudodifferential operator.

The definitions and needed facts of pseudodifferential calculus are gathered in Section 8. In this paper, we only consider symbols $a(x,\gamma,\tau,k)$ on $\mathbb{T}^d\times]0,+\infty[\times\mathbb{R}\times\mathbb{R}^d\backslash\{0\}$ with the quantization

$$(Op_a^\gamma)u(t,x) = \int_{\mathbb{Z}^d\times\mathbb{R}} e^{i(\tau t + k\cdot x)}a(x,\gamma,\tau,k)\hat{u}(\xi)\,d\xi$$

where $d\xi = dkd\tau$ and the measure on $\mathbb{Z}^d$ is the discrete measure.

Lemma 15. — *Let us set*

$$a(x,\zeta) = (2\pi)^d\int_0^{+\infty} e^{-(\gamma+i\tau)s}\, ik\cdot(\mathcal{F}_v\nabla_v f^0)(x,ks)\,ds, \tag{5.37}$$

for $\zeta = (\gamma,\tau,k) \in (0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\backslash\{0\}$, where again $\mathcal{F}_v$ stands for the Fourier transform in the v variable. Then, we have that

$$e^{-\gamma t}K_{\nabla_v f^0}(e^{\gamma t}h) = Op_a^\gamma(h), \quad \forall h \in \mathcal{S}.$$

Note that as usual when dealing with pseudodifferential calculus on the torus, we manipulate symbols defined in the whole space $\mathbb{R}^d$ in the k variable, though they are only used for $k \in \mathbb{Z}^d$ in the quantization formula.

Proof of Lemma 15. — Since h is 0 for negative times, we first note that

$$e^{-\gamma t} K_{\nabla_v f^0}(e^{\gamma t} h) = \int_{-\infty}^{t} e^{-\gamma(t-s)} \int \nabla_x h(s, x-(t-s)v) \cdot \nabla_v f^0(x,v)\, dvds.$$

By using the Fourier transform in x and t, we can write that

$$h(x,s) = \int_{\mathbb{Z}^d \times \mathbb{R}} e^{i(k\cdot x + \tau s)} \hat{h}(k,\tau)\, dkd\tau$$

with the convention that $\mathbb{Z}^d$ is equipped with the discrete measure dk. This yields

$$e^{-\gamma t} K_{\nabla_v f^0}(e^{\gamma t} h) =$$
$$\int_{\mathbb{Z}^d \times \mathbb{R}} e^{i(k\cdot x+\tau t)} \Big(\int_{-\infty}^{t} e^{-(\gamma+i\tau)(t-s)} \int e^{-ik\cdot v(t-s)} ik \cdot \nabla_v f^0(x,v)\, dv\, ds \Big) \hat{h}(k,\tau) dkd\tau,$$

that is,

$$e^{-\gamma t} K_{\nabla_v f^0}(e^{\gamma t} h) =$$
$$(2\pi)^d \int_{\mathbb{Z}^d \times \mathbb{R}} e^{i(k\cdot x+\tau t)} \Big(\int_{-\infty}^{t} e^{-(\gamma+i\tau)(t-s)} ik \cdot (\mathcal{F}_v \nabla_v f^0)(x, k(t-s))\, ds \Big) \hat{h}(k,\tau)\, dkd\tau.$$

Changing variable in the inside integral, we finally obtain

$$e^{-\gamma t} K_{\nabla_v f^0}(e^{\gamma t} h) = \int_{\mathbb{Z}^d \times \mathbb{R}} e^{i(k\cdot x+\tau t)} a(x,\zeta) \hat{h}(k,\tau)\, dkd\tau$$

with

$$a(x,\zeta) = (2\pi)^d \int_0^{+\infty} e^{-(\gamma+i\tau)s}\, ik \cdot (\mathcal{F}_v \nabla_v f^0)(x, ks)\, ds$$

as claimed.

□

Note that a does not actually depend on the time variable t. We shall now prove that a defined above is a good zero order symbol. The symbol seminorms are defined in Section 8, see (8.1), (8.2).

Lemma 16. — *Consider $a(x,\zeta)$, the symbol defined in* (5.37) *with $\zeta = (\gamma, \tau, k) = (\gamma, \xi)$, $\gamma > 0$ and take $\sigma > d/2$. Then we have that there exists $C_M > 0$ that depends only on M such that*

$$|a|_{M,0} \le C_M \|f^0\|_{\mathcal{H}^{2m}_{\sigma}}, \quad M \le 2m-3,$$
$$|a|_{M,1} \le C_M \|f^0\|_{\mathcal{H}^{2m}_{\sigma+2}}, \quad M \le 2m-4.$$

Moreover, a is homogeneous of degree zero:

$$a(x,\zeta) = a\left(x, \frac{\zeta}{\langle\zeta\rangle}\right), \quad \langle\zeta\rangle = (\gamma^2+\tau^2+|k|^2)^{\frac{1}{2}}.$$

Proof of Lemma 16. — Let us set $G(x,\eta) = (2\pi)^d(\mathcal{F}_v \nabla_v f^0)(x,\eta)$ so that
$$a(x,\zeta) = \int_0^{+\infty} e^{-(\gamma+i\tau)s}\, ik\cdot G(x,ks)\,ds.$$
and
$$\tilde{\zeta} = \frac{\zeta}{\langle\zeta\rangle}, \quad \langle\zeta\rangle = (\gamma^2+\tau^2+|k|^2)^{\frac12}.$$
Note that we have for $\sigma > d/2$ and every α, β, q, the estimate

(5.38)
$$|(\mathcal{F}_x\partial_x^\alpha\partial_\eta^\beta G)(l,\eta)| \lesssim \frac{1}{1+|\eta|^q}\left(\int_{\mathbb{R}^d}(1+|v|^2)^{\sigma+|\beta|}\,|(\mathcal{F}_x\nabla_v\partial_x^\alpha(I-\Delta_v)^{\frac{q}{2}}f^0)(l,v)|^2 dv\right)^{\frac12}.$$
By a change of variable $s = \tilde{s}/\langle\zeta\rangle$ in the integral defining a, we also easily observe that a is homogeneous of degree zero
$$a(x,\zeta) = a\left(x,\frac{\zeta}{\langle\zeta\rangle}\right).$$
Consequently, by using the definition of the symbol norms in the appendix, it suffices to prove that

$$\|\mathcal{F}_x(\partial_x^\alpha a)(\cdot,\tilde{\zeta})\|_{L^2(\mathbb{Z}^d,L^\infty(S_+))} \lesssim \|f^0\|_{\mathcal{H}^{2m}_\sigma}, \quad |\alpha|\le 2m-3, \tag{5.39}$$
$$\|(\mathcal{F}_x\partial_x^\alpha\nabla_{\tilde{\xi}}a)(\cdot,\tilde{\zeta})\|_{L^2(\mathbb{Z}^d,L^\infty(S_+))} \lesssim \|f^0\|_{\mathcal{H}^{2m}_{\sigma+1}}, \quad |\alpha|\le 2m-4. \tag{5.40}$$

where $S_+ = \{\tilde{\zeta} = (\tilde{\gamma},\tilde{\tau},\tilde{k}),\ \langle\tilde{\zeta}\rangle = 1,\ \tilde{\gamma}>0,\ \tilde{k}\neq 0\}$.

Since we have
$$(\mathcal{F}_x\partial_x^\alpha a)(l,\tilde{\zeta}) = \int_0^{+\infty} e^{-(\tilde{\gamma}+i\tilde{\tau})s}\, i\tilde{k}\cdot(\mathcal{F}_x\partial_x^\alpha)G(l,\tilde{k}s)\,ds,$$
by using (5.38) with $q=2$, and $\beta=0$, we obtain that
$$\begin{aligned}&|(\mathcal{F}_x\partial_x^\alpha a)(l,\tilde{\zeta})|\\ &\quad\lesssim \left(\int_{\mathbb{R}^d}(1+|v|^2)^\sigma\,|(\mathcal{F}_x\nabla_v\partial_x^\alpha(I-\Delta_v)f^0)(l,v)|^2dv\right)^{\frac12}\int_0^{+\infty}\frac{|\tilde{k}|}{1+|\tilde{k}|^2s^2}ds\\ &\quad\lesssim \left(\int_{\mathbb{R}^d}(1+|v|^2)^\sigma\,|(\mathcal{F}_x\nabla_v\partial_x^\alpha(I-\Delta_v)f^0)(l,v)|^2dv\right)^{\frac12}\int_0^{+\infty}\frac{1}{1+s^2}ds.\end{aligned}$$
This yields by using the Bessel identity
$$\|\mathcal{F}_x\partial_x^\alpha a\|_{L^2(\mathbb{Z}^d,L^\infty(\mathbb{S}_+))} \lesssim \|f^0\|_{\mathcal{H}^{|\alpha|+3}_\sigma}$$
and hence (5.39) is proven. Let us turn to the proof of (5.40). To estimate $\partial_x^\alpha\partial_{\tilde{\xi}}a$, we have to estimate the following two types of symbols
$$I_1^\alpha(x,\tilde{\zeta}) = \int_0^{+\infty} e^{-(\tilde{\gamma}+i\tilde{\tau})s}\, e_j\cdot\partial_x^\alpha G(x,\tilde{k}s)\,ds,$$
$$I_2^\alpha(x,\tilde{\zeta}) = \int_0^{+\infty} e^{-(\tilde{\gamma}+i\tilde{\tau})s}\, \tilde{k}s\cdot\partial_x^\alpha\partial_\eta^{\beta_1} G(x,\tilde{k}s)\,ds$$

where e_j is a unit vector and $|\beta_1| \leq 1$. For I_1^α, if $|\tilde{k}| \geq \frac{1}{2}$, we can proceed in the same way with (5.38) for $q = 2$, $\beta = 0$ and obtain

$$|\mathcal{F}_x I_1^\alpha(l, \tilde{\zeta})| \lesssim \left(\int_{\mathbb{R}^d} (1 + |v|^2)^\sigma \, |(\mathcal{F}_x \nabla_v \partial_x^\alpha (I - \Delta_v) f^0)(l, v)|^2 dv \right)^{\frac{1}{2}} \int_0^{+\infty} \frac{1}{1 + |\tilde{k}|^2 s^2} ds$$
$$\lesssim \left(\int_{\mathbb{R}^d} (1 + |v|^2)^\sigma \, |(\mathcal{F}_x \nabla_v \partial_x^\alpha (I - \Delta_v) f^0)(l, v)|^2 dv \right)^{\frac{1}{2}}$$

Note that for this argument, we use in a crucial way that $|\tilde{k}|$ is bounded from below. Otherwise since $\tilde{\zeta} \in S_+$, we have that $|\tilde{\gamma}|^2 + |\tilde{\tau}|^2 \geq \frac{3}{4}$ and consequently, we can integrate by parts in s in the integral to obtain that

$$|\mathcal{F}_x I_1^\alpha(l, \zeta)| \lesssim |(\mathcal{F}_x \partial_x^\alpha G)(l, 0)| + \int_0^{+\infty} |\tilde{k}| \, |(\mathcal{F}_x \partial_x^\alpha \nabla_\eta \, G)(l, \tilde{k}s) ds$$

and hence, by using again (5.38) with $q = 2$, and $|\beta| = 1$, we finally obtain that

$$\|\mathcal{F}_x I_1^\alpha\|_{L^2(\mathbb{Z}^d, L^\infty(\mathbb{S}_+))} \lesssim \|f^0\|_{\mathcal{H}^{2m}_{\sigma+1}}, \quad |\alpha| \leq 2m - 3.$$

To estimate I_2^α, we proceed as above: if $|\tilde{k}| \geq \frac{1}{2}$, we rely on (5.38) with $q = 3$ and $|\beta| \leq 1$, otherwise we use the same integration by parts argument together with (5.38) with $q = 2, 3$ and $|\beta| \leq 2$. We obtain

$$\|\mathcal{F}_x I_2^\alpha\|_{L^2(\mathbb{Z}^d, L^\infty(S_+))} \lesssim \|f^0\|_{\mathcal{H}^{2m}_{\sigma+2}}, \quad |\alpha| \leq 2m - 4.$$

This ends the proof.

□

We can now use symbolic calculus to estimate the solution of the integral equation (5.36).

Proposition 3. — *Consider h satisfying* (5.36), *assume that $2m > 4 + \frac{d}{2}$, that $2r > 2 + \frac{d}{2}$ and that for every $x \in \mathbb{T}^d$, the profile $f^0(x, \cdot)$ satisfies the c_0 Penrose stability criterion. Then there exists $\Lambda[\|f^0\|_{\mathcal{H}^{2m}_{2r}}]$ such that for every $\gamma \geq \Lambda[\|f^0\|_{\mathcal{H}^{2m}_{2r}}]$, we have the estimate*

$$\|h\|_{L^2(\mathbb{R} \times \mathbb{T}^d)} \lesssim \Lambda[\|f^0\|_{\mathcal{H}^{2m}_{2r}}] \|\mathcal{R}\|_{L^2(\mathbb{R} \times \mathbb{T}^d)}.$$

Proof of Proposition 3. — By using Lemma 15, we can write (5.36) under the form

$$h = Op_a^\gamma((I - \varepsilon^2 \Delta)^{-1} h) + \mathcal{R} = Op_{b^\varepsilon}^\gamma(h) + \mathcal{R}$$

with the symbol $b^\varepsilon(x, \zeta)$ defined by

$$b^\varepsilon(x, \zeta) = a(x, \zeta) \frac{1}{1 + \varepsilon^2 |k|^2}.$$

Note that this is exact since we are composing a pseudodifferential operator with a Fourier multiplier in the right order. Since a is homogeneous of degree zero in ζ, we have

$$b^\varepsilon(x, \zeta) = b(x, \varepsilon \zeta), \quad b(x, \zeta) = a(x, \zeta) \frac{1}{1 + |k|^2}$$

and thus $Op^{\gamma}_{b^{\varepsilon}} h = Op^{\varepsilon,\gamma}_{b}$ is a semiclassical pseudodifferential operator as defined in the Section 8. We thus have to study the equation

$$Op^{\varepsilon,\gamma}_{1-b}(h) = \mathcal{R}. \tag{5.41}$$

Thanks to Lemma 16, we have that $b \in S_{2m-3,0} \cap S_{2m-4,1}$. Moreover, we observe that

$$1 - b(x,\gamma,\tau,k) = \mathcal{P}(\gamma,\tau,k,f^0(x,\cdot))$$

and consequently, since f^0 satisfies the c_0 Penrose condition (1.5), we also get that $c = \frac{1}{1-b} \in S_{2m-3,0} \cap S_{2m-4,1}$. As a result, assuming that $2m > 4 + \frac{d}{2}$, we can find $M > d/2$ such that $c \in S_{M,1}$ and $1 - b \in S_{M+1,0}$ and moreover,

$$|c|_{M,1} + |1-b|_{M+1,0} \lesssim \Lambda[\|f^0\|_{\mathcal{H}^{2m}_{2r}}].$$

Consequently, by applying $Op^{\varepsilon,\gamma}_{c}$ to (5.41) and by using Proposition 7, we obtain that

$$\|h\|_{L^2(\mathbb{R}\times\mathbb{T}^d)} \lesssim \frac{1}{\gamma}\Lambda[\|f^0\|_{\mathcal{H}^{2m}_{2r}}]\|h\|_{L^2(\mathbb{R}\times\mathbb{T}^d)} + \Lambda[\|f^0\|_{\mathcal{H}^{2m}_{2r}}]\|\mathcal{R}\|_{L^2(\mathbb{R}\times\mathbb{T}^d)}.$$

The result follows by choosing γ sufficiently large. □

As a Corollary, we get an estimate for the solution of (5.34) on $[0,T]$.

Corollary 1. — *Consider $\tilde{h}$ the solution of* (5.36), *assume that $2m > 4 + \frac{d}{2}$, that $2r > 2 + \frac{d}{2}$ and that the c_0 Penrose criterion* (1.5) *is satisfied. Then there exists $\Lambda(\cdot,\cdot)$ such that the solution of* (5.34) *verifies the estimate*

$$\|\tilde{h}\|_{L^2([0,T],L^2_x)} \leq \Lambda[T, \|f^0\|_{\mathcal{H}^{2m}_{2r}}]\|\tilde{R}\|_{L^2([0,T],L^2_x)}.$$

Note that the assumption that $2m > 4 + \frac{d}{2}$ is satisfied if $2m > m_0$.

Proof of Corollary 1. — By using (5.35) and Proposition 3, we get that

$$\Big(\int_0^T e^{-2\gamma t}\|\tilde{h}(t,\cdot)\|^2_{L^2(\mathbb{T}^d)}dt\Big)^{\frac{1}{2}} \lesssim \|h\|_{L^2(\mathbb{R}\times\mathbb{T}^d)} \lesssim \Lambda[\|f^0\|_{\mathcal{H}^{2m}_{2r}}]\|\mathcal{R}\|_{L^2(\mathbb{R}\times\mathbb{T}^d)}.$$

Since we have taken $\mathcal{R}$ to be zero for $t \geq T$ and $t < 0$, we get by using again (5.35) that

$$\Big(\int_0^T e^{-2\gamma t}\|\tilde{h}(t,\cdot)\|^2_{L^2(\mathbb{T}^d)}dt\Big)^{\frac{1}{2}} \lesssim \Lambda[\|f^0\|_{\mathcal{H}^{2m}_{2r}}]\Big(\int_0^T e^{-2\gamma t}\|\tilde{R}(t,\cdot)\|^2_{L^2(\mathbb{T}^d)}dt\Big)^{\frac{1}{2}}.$$

Since γ was chosen as $\gamma = \Lambda[\|f^0\|_{\mathcal{H}^{2m}_{2r}}]$, the result follows by taking

$$\Lambda[T, \|f^0\|_{\mathcal{H}^{2m}_{2r}}] := \Lambda[\|f^0\|_{\mathcal{H}^{2m}_{2r}}]\exp\Big(\Lambda[\|f^0\|_{\mathcal{H}^{2m}_{2r}}]T\Big).$$

□

6. Proof of Theorem 1: conclusion

We are finally ready to close the bootstrap argument. For $2m > m_0$, $2r > r_0$, gathering the results of Lemma 13, Lemma 14 and Corollary 1, we get that for all $T \in [0, \min(T_0, \tilde{T}_0, \hat{T}_0, T^\varepsilon))$, for all $I, J \in \{1, \cdots, d\}^m$,

$$\|\partial_x^{\alpha(I,J)} \rho\|_{L^2([0,T],L^2_x)} \leq \Lambda(T, M_0) \left(M_0 + T^{\frac{1}{2}} \Lambda(T, R) \right),$$

and thus that

$$\|\rho\|_{L^2([0,T],H^{2m})} \leq \Lambda(T, M_0) \left(M_0 + T^{\frac{1}{2}} \Lambda(T, R) \right).$$

Using Lemma 4, we deduce the crucial estimate which is that

$$\mathcal{N}_{2m,2r}(T, f) \leq M_0 + T^{\frac{1}{2}} \Lambda(T, R) + \Lambda(T, M_0) \left(M_0 + T^{\frac{1}{2}} \Lambda(T, R) \right). \tag{6.1}$$

We can then easily conclude. Let us choose R large enough so that

$$\frac{1}{2} R > M_0 + \Lambda[0, M_0] M_0. \tag{6.2}$$

Now, R being fixed, we can choose by continuity $T^{\#} \in (0, \min(T_0, \tilde{T}_0, \hat{T}_0, T^\varepsilon)]$ such that for all $T \in [0, T^{\#}]$,

$$T^{\frac{1}{2}} \Lambda(T, R) + \Lambda[T, M_0] T^{\frac{1}{2}} \Lambda(T, R) + (\Lambda[T, M_0] - \Lambda[0, M_0]) M_0 < \frac{1}{2} R. \tag{6.3}$$

This entails that for all $T \in [0, T^{\#}]$, it is impossible to have $\mathcal{N}_{2m,2r}(T, f) = R$. Therefore, we deduce that $T^\varepsilon > T^{\#}$. We have thus proven that

$$\mathcal{N}_{2m,\,2r}(T, f) \leq R, \tag{6.4}$$

for some $T > 0$ and some $R > 0$, both independent of ε. To finish the proof of Theorem 1, it remains to check that the $c_0/2$ Penrose stability condition can be ensured. From the equation (1.1) and (6.4), we get that

$$\|\partial_t f\|_{L^\infty([0,T],\mathcal{H}^{2m-2}_{2r-1})} \leq \Lambda(T, R).$$

By using a Taylor expansion, we have that for all $t \in [0, T]$,

$$\int_0^{+\infty} e^{-(\gamma+i\tau)s} \, \frac{i\eta}{1+|\eta|^2} \cdot (\mathcal{F}_v \nabla_v f)(t, \eta s) \, ds = \int_0^{+\infty} e^{-(\gamma+i\tau)s} \, \frac{i\eta}{1+|\eta|^2} \cdot (\mathcal{F}_v \nabla_v f^0)(\eta s) \, ds + I(\gamma, \tau, \eta, t, x)$$

where $I(\gamma, \tau, \eta, x)$ satisfies the uniform estimate

$$|I(\gamma, \tau, \eta, t, x)| \leq CT \sup_{t \in [0,T]} \|\partial_t f\|_{\mathcal{H}^2_\sigma}$$

with $\sigma > d/2$. Since we have $2m \geq 4$ (by the assumption $2m > m_0$) and $2r > 1 + \frac{d}{2}$ this yields

$$|I(\gamma, \tau, \eta, t, x)| \leq T\Lambda(T, R).$$

Since f^0 satisfies the c_0 Penrose stability condition, it follows by taking a smaller time $T > 0$ if necessary, that for all $t \in [0,T]$ and all $x \in \mathbb{T}^d$, $f(t,x,\cdot)$ satisfies the $c_0/2$ Penrose condition.

7. Proofs of Theorems 2 and 3

The proofs of Theorems 2 and 3 will be intertwined since in order to get the convergence of Theorem 2 without extracting a subsequence, we shall need the uniqueness part of Theorem 3. Consequently, we shall first prove the uniqueness part of Theorem 3. The result will actually be a straightforward consequence of the following Proposition.

Proposition 4. — *We consider the following linear equation:*
(7.1)
$$\partial_t f + v\cdot\nabla_x f - \nabla_x \rho\cdot\nabla_v \overline{f} + E(t,x)\cdot\nabla_v f = F, \quad f_{|t=0} = f^0, \quad \rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,dv$$

where $E(t,x)$ is a given *vector field such that for some $T_0 > 0$, $E \in L^2((0,T_0), H^{2m})$ and $\overline{f}$ is a* given *function $\overline{f}(t,x,v) \in L^\infty(0,T_0], \mathcal{H}^{2m-1}_{2r}) \cap Lip([0,T_0], \mathcal{H}^{2m-2}_{2r-1})$ with $2m > m_0$, $2r > r_0$. Let us set*

$$R := \|\overline{f}\|_{L^\infty([0,T_0],\mathcal{H}^{2m-1}_{2r})} + \|\partial_t \overline{f}\|_{L^\infty([0,T_0],\mathcal{H}^{2m-2}_{2r-1})} + \|E\|_{L^2([0,T_0],H^{2m-1})}$$

and assume that for every x, the profile $\overline{f}(0,x,\cdot)$ satisfies the c_0 Penrose stability condition for some $c_0 > 0$. Then, there exists $T = T(c_0,R) \in (0,T_0]$ that depends only on c_0 and R such that for every $F \in L^2([0,T_0],\mathcal{H}^0_r)$ and $f^0 \in \mathcal{H}^0_r$, the solution f of (7.1) *satisfies the estimate*

$$\|\rho\|_{L^2([0,T]\times\mathbb{T}^d)} \leq \Lambda(\frac{1}{c_0}, R, T_0)\big(\|f^0\|_{\mathcal{H}^0_r} + \|F\|_{L^2([0,T_0],\mathcal{H}^0_r)}\big) \tag{7.2}$$

where $\Lambda(\frac{1}{c_0}, R, T)$ depends only on c_0, R and T_0.

Proof of Proposition 4. — The proof follows closely the analysis of equation (4.9) in the proof of Theorem 1 so that we shall only give the main steps. We first set $g(t,x,\Phi(t,x,v)) = f(t,x,v)$ with Φ being the solution of the Burgers equation (4.33) with initial data $\Phi(0,x,v) = v$, recall Lemma 10. Because of the regularity assumptions on E, Lemma 10 is still valid: such a smooth Φ exists on $[0,T(R)]$ for some $T(R) > 0$ and verifies the estimates (4.36), (4.37). We observe that g solves

$$\partial_t g + \Phi\cdot\nabla_x g - \nabla_x\rho(t,x)\cdot\nabla_v\overline{f}(t,x,\Phi) = F(t,x,\Phi) \tag{7.3}$$

and that

$$\int_{\mathbb{R}^d} g(t,x,v)J(t,x,v)\,dv = \rho(t,x)$$

with $J(t,x,v) = |\det \nabla_v \Phi(t,x,v)|$. To solve (7.3), we use the characteristics (5.2). Because of the previous estimates on Φ, the estimates of Lemma 12 are still valid. Proceeding as in the proof of Lemma 13, we can first obtain that

$$\rho(t,x) = K_H \rho + \int_0^t \int_{\mathbb{R}^d} F(s, X(s,t,x,v), \Phi(s,x,v)) J(t,x,v)\, dvds + \int_{\mathbb{R}^d} f^0(X(0,t,x,v), v) J(t,x,v)\, dv$$

where

$$H(t,s,x,v) = (\nabla_v \overline{f})(s, x-(t-s)v, \Psi(t,s,x,v)) J(t,x,\Psi(s,t,x,v)) \tilde{J}(t,s,x,v)$$

and $\tilde{J}(s,t,x,v) = |\det \nabla_v \Psi(s,t,x,v)|$. Again, by Taylor expanding H in time and by using Proposition 2 and remark 3 we obtain that

$$\rho(t,x) = K_{\nabla_v \overline{f}^0} \rho + \tilde{\mathcal{R}}$$

with the notation $\overline{f}^0(x,v) = \overline{f}(0,x,v)$ and where $\tilde{\mathcal{R}}$ is such that

$$\|\tilde{\mathcal{R}}\|_{L^2([0,T],L^2)} \leq \Lambda(T,R)(\|f^0\|_{\mathcal{H}^0_r} + \|F\|_{\mathcal{H}^0_r} + T^{\frac{1}{2}} \|\rho\|_{L^2([0,T],L^2)}) \tag{7.4}$$

for every $T \in [0, T(R)]$. In order to estimate the solution of the previous equation, we can again set $\rho = e^{\gamma t} h$, $\tilde{\mathcal{R}} = e^{\gamma t} \mathcal{R}$, assume that h and $\mathcal{R}$ are zero for $t < 0$ and that $\mathcal{R}$ is continued by zero for $t > T$. Then by using lemma 15, we end up with the equation

$$h = Op_a^\gamma h + \mathcal{R}$$

where a is still defined by (5.37) with f^0 replaced by $\overline{f}^0$. Because of the regularity assumptions on $\overline{f}$, the estimates of Lemma 16 are still verified. If a has the property that

$$|1 - a(x,\zeta)| \geq c_0, \quad \forall \zeta = (\gamma, \tau, k), \quad \gamma > 0,\ \tau \in \mathbb{R},\ k \in \mathbb{R}^d \backslash \{0\}, \tag{7.5}$$

then we can apply the operator $Op^\gamma_{\frac{1}{1-a}}$ and use Proposition 5 and Proposition 6 to get that for γ sufficiently large, we have

$$\|h\|_{L^2_{t,x}} \leq \Lambda(\frac{1}{c_0}, R) \|\mathcal{R}\|_{L^2_{t,x}}.$$

In view of (7.4), this yields that for every $T \in [0, T(R)]$ we have

$$\|\rho\|_{L^2([0,T],L^2)} \leq \Lambda(\frac{1}{c_0}, T, R)\big(\|f^0\|_{\mathcal{H}^0_r} + \|F\|_{\mathcal{H}^0_r} + T^{\frac{1}{2}} \|\rho\|_{L^2([0,T],L^2)}\big).$$

Consequently, if T is sufficiently small we get the estimate (7.2).

In order to finish the proof, we thus only have to check that the estimate (7.5) is verified. Let us recall that by definition of the Penrose stability condition, we have that for every $x \in \mathbb{T}^d$, the function

$$\mathcal{P}(\gamma, \tau, \eta, \overline{f}^0(x, \cdot)) = 1 - \int_0^{+\infty} e^{-(\gamma + i\tau)s} \frac{i\eta}{1 + |\eta|^2} \cdot (\mathcal{F}_v \nabla_v \overline{f}^0)(x, \eta s)\, ds,$$
$$\gamma > 0,\ \tau \in \mathbb{R},\ \eta \in \mathbb{R}^d \backslash \{0\},$$

verifies

$$\inf_{(\gamma,\tau,\eta) \in [0,+\infty) \times \mathbb{R} \times \mathbb{R}^d} |\mathcal{P}(\gamma, \tau, \eta, \overline{f}^0(x, \cdot))| \geq c_0.$$

Let us then define using polar coordinates the function $\tilde{\mathcal{P}}$ by

$$\tilde{\mathcal{P}}(\tilde{\gamma}, \tilde{\tau}, \tilde{\eta}, \sigma, \overline{f}^0(x, \cdot))$$
$$= \mathcal{P}(\sigma\tilde{\gamma}, \sigma\tilde{\tau}, \sigma\tilde{\eta}, \overline{f}^0(x, \cdot)), (\gamma, \tau, \eta) = \sigma(\tilde{\gamma}, \tilde{\tau}, \tilde{\eta}),\ r > 0,\ \tilde{\gamma} > 0,\ (\tilde{\gamma}, \tilde{\tau}, \tilde{\eta}) \in S_+$$

where $S_+ = \{(\tilde{\gamma}, \tilde{\tau}, \tilde{\eta}),\ \tilde{\gamma}^2 + \tilde{\tau}^2 + \tilde{\eta}^2 = 1,\ \tilde{\gamma} > 0, \tilde{\eta} \neq 0\}$. Note that we have

$$\tilde{\mathcal{P}}(\tilde{\gamma}, \tilde{\tau}, \tilde{\eta}, \sigma, \overline{f}^0) = 1 - \int_0^{+\infty} e^{-(\tilde{\gamma} + i\tilde{\tau})s} \frac{i\tilde{\eta}}{1 + \sigma^2 |\tilde{\eta}|^2} \cdot (\mathcal{F}_v \nabla_v \overline{f}^0)(x, \tilde{\eta} s)\, ds.$$

If $\overline{f}^0 \in \mathcal{H}^2_r$, the function $\tilde{\mathcal{P}}(\cdot, \overline{f}^0)$ can be extended as a continuous function on $S_+ \times [0 + \infty[$. The Penrose stability condition thus implies $\tilde{\mathcal{P}} \geq c_0$ on $S_+ \times [0, +\infty[$. In particular for $\sigma = 0$, we observe that

$$\tilde{\mathcal{P}}(\tilde{\gamma}, \tilde{\tau}, \tilde{\eta}, 0, \mathbf{f}) = 1 - a(\tilde{\gamma}, \tilde{\tau}, \tilde{\eta}).$$

We thus obtain that $|1 - a| \geq c_0$ on S_+. Since a is homogeneous of degree zero, this yields that (7.5) is verified. This ends the proof. □

As an immediate corollary of the previous proposition, we get an uniqueness property for the limit equation (1.3).

Corollary 2. — *Let $f_1, f_2 \in \mathcal{C}([0,T], \mathcal{H}^{2m-1}_{2r})$ with $2m > m_0$, $2r > r_0$ be two solutions of (1.3) with the same initial condition f^0. Setting $\rho_i := \int f_i\, dv$, we assume that $\rho_1, \rho_2 \in L^2([0,T], H^{2m})$. Assume that furthermore, there is $c_0 > 0$ such that f_1 is such that $v \mapsto f_1(t,x,v)$ satisfies the c_0 Penrose condition for every $t \in [0,T]$ and $x \in \mathbb{T}^d$. Then we have that $f_1 = f_2$ on $[0,T] \times \mathbb{T}^d \times \mathbb{R}^d$.*

Proof of Corollary 2. — Let

$$R = \max_{i=1,2} \left(\|f_i\|_{L^\infty([0,T], \mathcal{H}^{2m-1}_{2r})} + \|\rho_i\|_{L^2([0,T], H^{2m})} \right).$$

We set $f = f_1 - f_2$, and observe that f satisfies the equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \rho \cdot \nabla_v f_1 - \nabla_x \rho_2 \cdot \nabla_v f = 0, \quad f|_{t=0} = 0, \tag{7.6}$$

where $\rho := \int f\, dv$. We are thus in the framework of Proposition 4 with $E = -\nabla_x \rho_2$, $\overline{f} = f_1$ and zero data (that it say $F = 0$ and zero initial data). Moreover, we observe that thanks to the equation (1.3), we also have that

$$\|\partial_t f_i\|_{L^\infty([0,T], \mathcal{H}^{2m-2}_{2r-1})} \leq \Lambda(T, R).$$

We are thus in the framework of Proposition 4. From (7.2), we deduce that there exists $T(c_0, R)$ such that $\rho = 0$ in $[0, T(c_0, R)]$. This yields that on $[0, T(c_0, R)]$, f satisfies the homogeneous transport equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \rho_2 \cdot \nabla_v f = 0$$

with zero initial data and thus $f = 0$ on $[0, T(c_0, R)]$. We can then apply again Proposition 4 starting from $T(c_0, R)$ (which is valid since $f_1(T(c_0, R), \cdot)$ still satisfies the c_0 Penrose stability condition). Since the estimate (7.2) is valid on an interval of time that depends only on R and c_0, we then obtain that $f = 0$ on $[0, 2T(c_0, R)]$. Repeating the argument, we finally obtain after a finite number of steps that $f = 0$ on $[0, T]$. This ends the proof.

□

7.1. Proof of Theorem 2. — We start by applying Theorem 1 to get $T, R > 0$ independent of ε, such that $f_\varepsilon \in \mathcal{C}([0, T], \mathcal{H}^{2m}_{2r})$ satisfies (1.1) with

$$\sup_{\varepsilon \in (0, \varepsilon_0]} \mathcal{N}_{2m,2r}(T, f_\varepsilon) \leq R.$$

We can now use standard compactness arguments to justify the quasineutral limit: f_ε is uniformy bounded in $\mathcal{C}([0, T], \mathcal{H}^{2m-1}_{2r})$ and from (1.1), we get that $\partial_t f_\varepsilon$ is uniformly bounded in $L^\infty([0, T], \mathcal{H}^{2m-2}_{2r-1})$. Consequently, by Ascoli Theorem there exists $f \in \mathcal{C}([0, T], L^2)$ and a sequence ε_n such that f_{ε_n} converges to f in $\mathcal{C}([0, T], L^2_{x,v})$. By interpolation, we also actually have convergence in $\mathcal{C}([0, T], \mathcal{H}^{2m-1-\delta}_{2r-\delta})$ for every $\delta > 0$. By Sobolev embedding this yields in particular that f_{ε_n} converges to f in $L^\infty([0, T] \times \mathbb{T}^d \times \mathbb{R}^d)$ and that ρ_ε converges to $\rho = \int_{\mathbb{R}^d} f \, dv$ in $L^2([0, T], L^2) \cap L^\infty([0, T] \times \mathbb{T}^d)$. From these strong convergences, we easily obtain that f is solution of (1.3) and that f satisfies the $c_0/2$ Penrose stability condition on $[0, T]$. Moreover, by standard weak-compactness arguments, we also easily obtain that $f \in L^\infty([0, T], \mathcal{H}^{2m-1}_{2r}) \cap \mathcal{C}_w([0, T], \mathcal{H}^{2m-1}_{2r})$ (that is to say continuous in time with $\mathcal{H}^{2m-1}_{2r}$ equipped with the weak topology) and that $\rho \in L^2([0, T], H^{2m})$. With this regularity of ρ, we can then deduce by standard arguments from the energy estimate for (1.3) (which is just (3.11) with $E = -\nabla_x \rho$) that $f \in \mathcal{C}([0, T], \mathcal{H}^{2m-1}_{2r})$.

Thanks to the uniqueness for (1.3) proved in Corollary 2, we can get by standard arguments that we actually have the full convergence of f_ε to f and not only the subsequence f_{ε_n}.

7.2. Proof of Theorem 3. — With the choice $f^0_\varepsilon = f_0$ for all $\varepsilon \in (0, 1]$, Theorem 2 provides the existence part. The uniqueness is a consequence of Corollary 2 and the fact that f satisfies the $c_0/2$ Penrose stability condition (1.5) for every $t \in [0, T]$ and $x \in \mathbb{T}^d$.

8. Pseudodifferential calculus with parameter

In this section we shall prove the basic results about pseudodifferential calculus that we need in our proof. For more complete statements and results, we refer for example to [**34, 35**]. We consider symbols $a(x,\gamma,\tau,k)$ on $\mathbb{T}^d\times]0,+\infty[\times\mathbb{R}\times\mathbb{R}^d\backslash\{0\}$, $\gamma>0$ has to be thought to as a parameter. We set $\zeta=(\gamma,\tau,k)$ and $\xi=(\tau,k)\in\mathbb{R}\times\mathbb{R}^d\backslash\{0\}$. Note that we do not need to include a dependence on the time variable t in our symbols (so that we actually consider Fourier multipliers in the time variable). We use the quantization

$$(Op_a^\gamma)u(t,x)=\int_{\mathbb{Z}^d\times\mathbb{R}} e^{i(\tau t+k\cdot x)}a(x,\zeta)\hat{u}(\xi)\,d\xi$$

where $d\xi=dkd\tau$ and the measure on $\mathbb{Z}^d$ is the discrete measure. The Fourier transform $\hat{u}$ is defined as

$$\hat{u}(\xi)=(2\pi)^{-(d+1)}\int_{\mathbb{T}^d\times\mathbb{R}} e^{-i(\tau t+k\cdot x)}u(t,x)dtdx.$$

We introduce the following seminorms of symbols:

$$|a|_{M,0}=\sup_{|\alpha|\leq M}\|\mathcal{F}_x(\partial_x^\alpha a)\|_{L^2(\mathbb{Z}^d,L^\infty_\zeta)},\tag{8.1}$$

$$|a|_{M,1}=\sup_{|\alpha|\leq M}\|\langle\zeta\rangle\mathcal{F}_x(\partial_x^\alpha\nabla_\xi a)\|_{L^2(\mathbb{Z}^d,L^\infty_\zeta)}.\tag{8.2}$$

where

$$\langle\zeta\rangle=(\gamma^2+\tau^2+|k|^2)^{\frac{1}{2}}.$$

We shall say that $a\in S_{M,0}$ if $|a|_{M,0}<+\infty$ and $a\in S_{M,1}$ if $|a|_{M,1}<+\infty$. The use of these seminorms compared to some more classical ones will allow us to avoid to lose too many derivatives while keeping very simple proofs.

Note that we can easily relate $|a|_{M,0}$ to more classical symbol seminorms up to loosing more derivatives. For example, we have for every $M\geq 0$

$$\sup_{|\alpha|\leq M}\sup_{x,\,\zeta}|\partial_x^\alpha a(x,\zeta)|\lesssim|a|_{M+s,0}$$

with $s>d/2$. The following results refine slightly in terms of the regularity of the symbols, the classical results of L^2 continuity for symbols in $S^0_{0,0}$ that are compactly supported in x, see for example [**39**].

Proposition 5. — *Assume that $M>d/2$ and that $a\in S_{M,0}$. Then, there exists $C>0$ such that for every $\gamma>0$*

$$\|Op_a^\gamma u\|_{L^2(\mathbb{R}\times\mathbb{T}^d)}\leq C|a|_{M,0}\|u\|_{L^2(\mathbb{R}\times\mathbb{T}^d)}.$$

Proof of Proposition 5. — In the following $\lesssim$ means $\leq C$ with C that does not depend on γ. We can write

$$\begin{aligned} Op_a^\gamma u(t,x) &= -\int_{\mathbb{Z}^d} e^{ix\cdot k'}\Big(\int_{\mathbb{Z}^d\times\mathbb{R}} e^{i(\tau t+k\cdot x)}\mathcal{F}_x a(k',\zeta)\hat{u}(\xi)\,d\xi\Big)dk' \\ &= \int_{\mathbb{Z}^d\times\mathbb{R}} e^{i(\tau t+l\cdot x)}\Big(\int_{\mathbb{Z}^d}\mathcal{F}_x a(k-l,\gamma,\tau,k)\hat{u}(\tau,k)dk\Big)d\tau dl \end{aligned}$$

and hence we obtain from the Bessel identity that

$$\|Op_a^\gamma u\|_{L^2(\mathbb{R}\times\mathbb{T}^d)} \lesssim \left\|\left\|\int_{\mathbb{Z}^d}\mathcal{F}_x a(k-\cdot,\gamma,\tau,k)\hat{u}(\tau,k)dk\right\|_{L^2(\mathbb{Z}^d)}\right\|_{L^2(\mathbb{R}_\tau)}.$$

By using Cauchy-Schwarz and Fubini, we get in a classical way that

$$\begin{aligned} &\left\|\int_{\mathbb{Z}^d}\mathcal{F}_x a(k-\cdot,\gamma,\tau,k)\hat{u}(\tau,k)dk\right\|^2_{L^2(\mathbb{Z}^d)} \\ &\lesssim \|\sup_k|\mathcal{F}_x a(\cdot,\gamma,\tau,k)|\,\|_{L^1(\mathbb{Z}^d)}\int_{\mathbb{Z}^d\times\mathbb{Z}^d}|\mathcal{F}_x a(k-l,\gamma,\tau,k)|\,|\hat{u}(\tau,k)|^2dkdl \\ &\lesssim \|\sup_k|\mathcal{F}_x a(\cdot,\gamma,\tau,k)|\,\|_{L^1(\mathbb{Z}^d)}\sup_k\|\mathcal{F}_x a(\cdot,\gamma,\tau,k)\|_{L^1(\mathbb{Z}^d)}\|\hat{u}(\tau,\cdot)\|^2_{L^2(\mathbb{Z}^d)} \\ &\lesssim \|\sup_k|\mathcal{F}_x a(\cdot,\gamma,\tau,k)|\,\|^2_{L^1(\mathbb{Z}^d)}\|\hat{u}(\tau,\cdot)\|^2_{L^2(\mathbb{Z}^d)}. \end{aligned}$$

By integrating in time, we thus obtain that

$$\|Op_a^\gamma u\|_{L^2(\mathbb{R}\times\mathbb{T}^d)} \lesssim \|\mathcal{F}_x a\|_{L^1(\mathbb{Z}^d,L^\infty_\zeta)}\|u\|_{L^2(\mathbb{R}\times\mathbb{T}^d)}.$$

To conclude, it suffices to notice that

$$\|\mathcal{F}_x a\|_{L^1(\mathbb{Z}^d,L^\infty_\zeta)} \lesssim |a|_{M,0}$$

for $M > d/2$.

□

We shall now state a result of symbolic calculus.

Proposition 6. — *Assume that $a\in S_{M,1}$ and that $b\in S_{M+1,0}$ with $M>d/2$. Then there exists $C>0$ such that for every $\gamma>0$, we have*

$$\|Op_a^\gamma Op_b^\gamma(u) - Op_{ab}^\gamma(u)\| \leq \frac{C}{\gamma}|a|_{M,1}|b|_{M+1,0}\,\|u\|_{L^2(\mathbb{R}\times\mathbb{T}^d)}.$$

Note that the above estimate is especially useful for large γ since the right hand side can be made small by taking γ sufficiently large.

Proof of Proposition 6. — Note that for $a\in S_{M,0}$, $b\in S_{M,0}$ and $M>d/2$, we have by elementary convolution estimates that

$$|ab|_{M,0} \lesssim |a|_{M,0}\|\mathcal{F}_x b\|_{L^1(\mathbb{Z}^d,L^\infty_\zeta)} + |b|_{M,0}\|\mathcal{F}_x a\|_{L^1(\mathbb{Z}^d,L^\infty_\zeta)} \lesssim |a|_{M,0}|b|_{M,0}$$

and thus that $ab \in S_{M,0}$. This yields that Op^γ_{ab} is a well-defined continuous operator on L^2 thanks to Proposition 5. Next, using the usual formulas for pseudodifferential operators, we find that

$$Op^\gamma_a Op^\gamma_b = Op^\gamma_c$$

with c given by

$$c(x,\zeta) = \int_{\mathbb{Z}^d} e^{ik'\cdot x} a(x,\gamma,\tau,k+k')\mathcal{F}_x b(k',\zeta)\, dk', \quad \zeta = (\gamma,\tau,k).$$

We thus get that

$$c(x,\zeta) - a(x,\zeta)b(x,\zeta) = \int_{\mathbb{Z}^d} e^{ik'\cdot x} \int_0^1 \nabla_k a(x,\gamma,\tau,k+sk')\, ds \cdot k'\, \mathcal{F}_x b(k',\zeta)\, dk' \tag{8.3}$$

$$=: \frac{1}{\gamma} d(x,\zeta). \tag{8.4}$$

By using Proposition 5, we can just prove that $d \in S_{M,0}$ for $M > d/2$ and estimate its norm. By taking the Fourier transform in x, we obtain that

$$(\mathcal{F}_x \partial_x^\alpha d)(l,\gamma,\tau,k) = \gamma \int_0^1 \int_{\mathbb{Z}^d} (il)^\alpha (\mathcal{F}_x \nabla_k a)(l-k',\gamma,\tau,k+sk') \cdot k' \mathcal{F}_x b(k',\zeta)\, dk' ds.$$

This yields

$$\begin{aligned}
&\|(\mathcal{F}_x \partial_x^\alpha d)(l,\gamma,\cdot)\|_{L^\infty_\zeta} \\
&\qquad \lesssim \int_{\mathbb{Z}^d} |l-k'|^{|\alpha|}\, \|\, |(\gamma,\cdot)| (\mathcal{F}_x \nabla_k a)(l-k',\gamma,\cdot)\|_{L^\infty_\zeta}\, |k'|\, \|\mathcal{F}_x b(k',\gamma,\cdot)\|_{L^\infty_\zeta}\, dk' \\
&\qquad\quad + \int_{\mathbb{Z}^d} \|\, |(\gamma,\cdot)| (\mathcal{F}_x \nabla_k a)(l-k',\gamma,\cdot)\|_{L^\infty_\zeta}\, |k'|^{|\alpha|+1}\, \|\mathcal{F}_x b(k',\gamma,\cdot)\|_{L^\infty_\zeta}\, dk'.
\end{aligned}$$

From standard convolution estimates, we obtain that

$$|d|_{M,0} \lesssim \left(|a|_{M,1}\, \||k|\mathcal{F}_x \nabla_x b\|_{L^1(\mathbb{Z}^d, L^\infty_\zeta)} + |b|_{M+1,0}\, \|\, |(\gamma,\cdot)| \mathcal{F}_x \nabla_k a\|_{L^1(\mathbb{Z}^d, L^\infty_\zeta)} \right)$$

and thus, for $M > d/2$, we finally get that

$$|d|_{M,0} \lesssim |a|_{M,1} |b|_{M+1,0}.$$

From Proposition 5, we get that

$$\|Op^\gamma_d u\|_{L^2(\mathbb{R}\times\mathbb{T}^d)} \lesssim |a|_{M,1} |b|_{M+1,0} \|u\|_{L^2(\mathbb{R}\times\mathbb{T}^d)}.$$

Since by definition of d, we have $Op^\gamma_a Op^\gamma_b - Op^\gamma_{ab} = \frac{1}{\gamma} Op^\gamma_d$, the result follows.

□

We shall finally define a semiclassical version of the above calculus. For any symbol $a(x,\zeta)$ as above, we set for $\varepsilon \in (0,1]$, $a^\varepsilon(x,\zeta) = a(x,\varepsilon\zeta) = a(x,\varepsilon\gamma,\varepsilon\tau,\varepsilon k)$ and we define for $\gamma \geq 1$,

$$(Op^{\varepsilon,\gamma}_a u)(t,x) = (Op^\gamma_{a^\varepsilon} u)(t,x). \tag{8.5}$$

For this calculus, we have the following result:

Proposition 7. — *There exists $C > 0$ such that for every $\varepsilon \in (0,1]$ and for every $\gamma \geq 1$ we have*

– *for every $a \in S_{M,0}$ with $M > d/2$,*

$$\|Op_a^{\varepsilon,\gamma} u\|_{L^2(\mathbb{R}\times\mathbb{T}^d)} \leq C|a|_{M,0}\|u\|_{L^2(\mathbb{R}\times\mathbb{T}^d)},$$

– *for every $a \in S_{M,1}$ and for every $b \in S_{M+1,0}$, with $M > d/2$,*

$$\|Op_a^{\varepsilon,\gamma} Op_b^{\varepsilon,\gamma}(u) - Op_{ab}^{\varepsilon,\gamma}(u)\| \leq \frac{C}{\gamma}|a|_{M,1}|b|_{M+1,0}\,\|u\|_{L^2(\mathbb{R}\times\mathbb{T}^d)}.$$

Proof of Proposition 8.5. — The proof is a direct consequence of Proposition 5 and Proposition 6 since for any symbol a, we have by definition of a^ε that

$$|a^\varepsilon|_{M,0} = |a|_{M,0}, \quad |a^\varepsilon|_{M,1} = |a|_{M,1}.$$

□

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