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TREND TO EQUILIBRIUM AND SPECTRAL LOCALIZATION PROPERTIES FOR THE LINEAR BOLTZMANN EQUATION

by

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Abstract. The aim of this note is to present the results from [**11, 12**], which deal with the linear Boltzmann equation, set in a bounded domain and in the presence of an external force. A specificity of these works is that the collision operator is allowed to be degenerate in the following two senses: (1) the associated collision kernel may vanish in a large subset of the phase space; (2) we do not assume that it is bounded below by a Maxwellian at infinity in velocity.

We study:

– the large time behavior of solutions of the linear Boltzmann equation, by giving criteria (inspired from control theory) which ensure converge towards an equilibrium and when possible, convergence at an exponential rate [**11**] ;
– some properties of localization for the spectrum of the associated operator [**12**].

Résumé **(Relaxation vers l'équilibre et propriétés de localisation spectrale pour l'équation de Boltzmann linéaire)**

L'objectif de cette note est de présenter les résultats de [**11, 12**], qui concernent l'équation de Boltzmann linéaire, posée dans un domaine borné et en présence d'une force extérieure. Une spécificité de ces travaux réside dans la prise en compte d'opérateurs de collision dégénérés aux deux sens suivants : (1) le noyau de collision associé peut s'annuler sur un grand sous-ensemble de l'espace des phases ; (2) le noyau de collision n'est pas supposé être minoré par une Maxwellienne à l'infini en vitesse.

Nous étudions :

– le comportement en temps grand des solutions l'équation de Boltzmann linéaire, en donnant des critères (inspirés par la théorie du contrôle) pour assurer la convergence vers un équilibre et quand cela est possible, convergence à un taux exponentiel [**11**] ;
– les propriétés de localisation du spectre de l'opérateur associé [**12**].

1. Introduction

We study the **linear Boltzmann equation**

$$\partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f = \int_{\mathbb{R}^d} [k(x, v', v) f(v') - k(x, v, v') f(v)]\, dv', \tag{1.1}$$

for $(t, x, v) \in \mathbb{R}^+ \times \mathbb{T}^d \times \mathbb{R}^d$ and $d \in \mathbb{N}^*$. We focus here on this simple geometric setting in order to present our results, but several generalizations (including the case of a bounded domain of $\mathbb{R}^d$ with specular reflection, or a general Riemannian framework) are actually provided in [**11, 12**].

The equation (1.1) is a classical model from statistical physics, and has applications in several domains, including neutronics, radiative transfer, or rarefied gases. It allows to describe the dynamics of a population of particles, *via* the study of the evolution of their so-called **distribution function** $f(t,x,v) \geq 0$. This quantity can be seen as the density at time t of particles with position x and velocity v.

The dynamics of these particles is dictated by two effects of different nature:

– **Transport** is driven by the **hamiltonian** $H(x,v) = \frac{|v|^2}{2} + V(x)$, where V is a smooth potential, i.e. $V \in W^{2,\infty}(\mathbb{T}^d)$. Loosely speaking, this corresponds to the vector field in the the left-hand side of (1.1);

– **Collisions** (to be understood here as an interaction with a fixed background) are described by the **collision kernel** $k(x,v,v') \in C^0(\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d)$. Loosely speaking, this corresponds to the integral operator in the right-hand side of (1.1).

Two important properties of (1.1) are the **conservation of the total mass**

$$\forall t \geq 0, \quad \frac{d}{dt}\int_{\mathbb{T}^d\times\mathbb{R}^d} f(t,x,v)\,dvdx = 0,$$

and the **maximum principle**

$$f|_{t=0} \geq 0 \implies \forall t \geq 0, f(t) \geq 0.$$

The linear Boltzmann equation is a prototype of an **hypocoercive** PDE (in the sense of Gallay-Villani, [**22**]). Hypocoercivity refers to the situation where an interplay between a conservative part (transport) and a "degenerate" dissipative part (collisions) leads to **convergence to equilibrium**, while each part on its own is not sufficient to guarantee this convergence.

Consider for instance the following simple linear Boltzmann equation:

$$\partial_t f + v\cdot\nabla_x f = \sigma\left[\left(\int_{\mathbb{S}^{d-1}} f\,dv\right) - f\right], \tag{1.2}$$

for $(t,x,v) \in \mathbb{R}^+ \times \mathbb{T}^d \times \mathbb{S}^{d-1}$, $d \in \mathbb{N}^*$, where σ is **positive** constant. We have the following classical result.

Theorem 1.1 **(Ukai, Point, Ghidouche [19]).** *There exists $C,\gamma > 0$ such that for any $f_0 \in L^1$,*

$$\forall t \geq 0, \quad \left\| f(t) - \int_{\mathbb{T}^d\times\mathbb{S}^{d-1}} f_0\,dv\,dx \right\|_{L^1} \leq Ce^{-\gamma t}\left\| f_0 - \int_{\mathbb{T}^d\times\mathbb{S}^{d-1}} f_0\,dv\,dx \right\|_{L^1}. \tag{1.3}$$

where $f(t)$ is the solution of (1.2) *with initial datum f_0.*

Whenever an inequality of the form (1.3) holds, we say that **uniform/exponential convergence to equilibrium** holds.

In [**5, 6**], Desvillettes and Villani initiated a program in order to study such phenomena for a wide class of kinetic equations. They introduced a method to prove convergence to the (Maxwellian) equilibrium with a rate which is faster than any polynomial for:

– the (linear) Fokker-Planck equation [**5**];
– the (nonlinear) Boltzmann equation, assuming *a priori* estimates on the solution [**6**].

Subsequently, Hérau and Nier [**14**], Hérau [**13**], Mouhot and Neumann [**17**], Villani [**22**], Dolbeault, Mouhot and Schmeiser [**8**], among others, proposed several methods to study various kinetic models (in particular a potential $V \neq 0$ is allowed).

In all these references, for what concerns the linear Boltzmann equation, the collision kernels are assumed to satisfy a kind of **non-degeneracy property**. Denoting by

$$\mathcal{M}(v) := \frac{1}{(2\pi)^{d/2}} e^{-\frac{|v|^2}{2}}$$

the **Maxwellian** equilibrium, the result of [**8**] on this equation may be stated as follows:

$$\exists \lambda > 0, \forall (x, v, v') \in \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d, \ \frac{k(x, v, v')}{\mathcal{M}(v')} \geq \lambda$$
$$\implies \text{ exponential convergence to equilibrium in a weighted } L^2 \text{ space.}$$

More recently, as opposed to the previous situation, Bernard and Salvarani [**3, 2**] (see also Desvillettes and Salvarani [**4**]) consider the following **degenerate** case.

$$\partial_t f + v \cdot \nabla_x f = \sigma(x) \left[\left(\int_{\mathbb{S}^{d-1}} f \, dv \right) - f \right]$$

for $(t, x, v) \in \mathbb{R}^+ \times \mathbb{T}^d \times \mathbb{S}^{d-1}$, $d \in \mathbb{N}^*$, where $\sigma \geq 0$ may vanish on a subset of $\mathbb{T}^d$. They prove that exponential decay is equivalent to the fact that $\{\sigma > 0\}$ satisfies a **geometric control condition**, in the spirit of Bardos, Lebeau, Rauch, Taylor [**18, 1**]. We shall come back to this point later.

Let us now list the precise assumptions we make on the collision kernel k in (1.1) (see [**11, 12**]).

- **A1.** The collision kernel k is non-negative on $\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d$.
- **A2.** The Maxwellian distribution $\mathcal{M}(v)$ cancels the collision operator, i.e.

$$\forall (x, v) \in \mathbb{T}^d \times \mathbb{R}^d, \quad \int_{\mathbb{R}^d} [k(x, v', v)\mathcal{M}(v') - k(x, v, v')\mathcal{M}(v)] \, dv' = 0.$$

  Note that this assumption entails in particular that $(x, v) \mapsto e^{-V(x)}\mathcal{M}(v)$ is a stationary solution of the linear Boltzmann equation (and hence generates a one dimensional vector space of stationary solutions).
- **A3.** $\tilde{k}(x, v', v) := \frac{k(x,v',v)}{\mathcal{M}(v)} \in L^\infty(\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d)$.

  We actually handle some $\tilde{k}$ which are for instance polynomially growing in the v and v' variables, see [**11**], but we stick to this simple assumption for the sake of clarity of exposition.

Therefore, we see that within these assumptions, the collision kernel can be degenerate in the following two senses:

– It can **vanish in a large subset** of the phase space $\mathbb{T}^d \times \mathbb{R}^d$;

– The function $\tilde{k}$ is **not** assumed to be **bounded below** by a fixed positive constant at infinity.

Our goals are to find **geometric criteria** (on the hamiltonian H and the collision kernel k) to characterize:

- **Q1** convergence to an equilibrium,
- **Q1'** exponential convergence to this equilibrium.

This gives rise to a natural related problem:

- **Q2** describe the structure and the localization properties of the spectrum of the associated Boltzmann operator.

The work **[11]** is dedicated to **Q1–Q1'**, while **[12]** tackles **Q2**. We give an answer to these questions with a point of view inspired by **control theory**.

The remainder of this note is organized as follows. We start by introducing in Section 2 several geometric definitions needed to formulate our results. Then Sections 3 and 4 are respectively concerned with **Q1** and **Q1'**. Finally we address **Q2** in Section 5 and describe some localization properties of the spectrum of the associated operator.

2. Geometric definitions

Two natural "objects" related to transport and collisions are respectively the **characteristics** and the **set where collisions are effective**, which we introduce now.

Definition 2.1. *The characteristics associated to the hamiltonian $H(x,v) = \frac{|v|^2}{2} + V(x)$ is the family of diffeomorphisms $(\phi_t)_{t\geq 0}$ defined for all $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$ by $\phi_t(x,v) := (X_t(x,v), \Xi_t(x,v))$ with*

$$\begin{cases} \dfrac{dX_t(x,v)}{dt} = \Xi_t(x,v), \quad \dfrac{d\Xi_t(x,v)}{dt} = -\nabla_x V(X_t(x,v)), \\ X_{t=0} = x, \quad \Xi_{t=0} = v. \end{cases}$$

Definition 2.2. *The open set ω of $\mathbb{T}^d \times \mathbb{R}^d$ where* **collisions are effective** *is*

$$\begin{aligned} \omega : &= \left\{ (x,v) \in \mathbb{T}^d \times \mathbb{R}^d, \int_{\mathbb{R}^d} k(x,v,v')\, dv' > 0 \right\} \\ &= \left\{ (x,v) \in \mathbb{T}^d \times \mathbb{R}^d, \exists v' \in \mathbb{R}^d, k(x,v,v') > 0 \right\} \\ &= \left\{ (x,v) \in \mathbb{T}^d \times \mathbb{R}^d, \exists v' \in \mathbb{R}^d, k(x,v',v) > 0 \right\}. \end{aligned}$$

The last two equalities are consequences of the non-negativity of k and of Assumption **A2**.

Now, in order to understand how transport and collisions interact, we introduce:

– several **Geometric Control Conditions**;

– a structural-geometric definition, involving a relevant **equivalence relation**.

2.1. Geometric control conditions. In this section, U denotes an open subset of $\mathbb{T}^d \times \mathbb{R}^d$.

We start with the classical **Geometric Control Condition (GCC for short)** of Bardos, Lebeau, Rauch, Taylor [**18, 1**]. We then introduce an "almost everywhere GCC", and recall the definition of the Lebeau constants [**16**].

Definition 2.3. *We say that U satisfies the Geometric Control Condition with respect to H if there exists $T > 0$ such that, for any $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$, there is $t \in [0, T]$ with $\phi_t(x, v) \in U$.*

Definition 2.4. *We say that U satisfies the* **almost everywhere infinite time (aeit) Geometric Control Condition** *with respect to H if for almost all $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$, there is $t \in [0, +\infty)$ with $\phi_t(x, v) \in U$.*

Definition 2.5. *The* **Lebeau constants** *(introduced in* [**16**] *in the context of the damped wave equation) are defined as follows:*

$$C^-(\infty) := \sup_{T \in \mathbb{R}^+} \inf_{(x,v) \in \mathbb{T}^d \times \mathbb{R}^d} \frac{1}{T} \int_0^T \left(\int_{\mathbb{R}^d} k(\phi_t(x, v), v')\, dv' \right) dt,$$

$$C^+(\infty) := \inf_{T \in \mathbb{R}^+} \sup_{(x,v) \in \mathbb{T}^d \times \mathbb{R}^d} \frac{1}{T} \int_0^T \left(\int_{\mathbb{R}^d} k(\phi_t(x, v), v')\, dv' \right) dt.$$

Because of the lack of compactness of the phase-space $\mathbb{T}^d \times \mathbb{R}^d$, it turns out that GCC is not adapted to our needs. This leads to the introduction of the slighlty stronger geometric control condition $C^-(\infty) > 0$. The aeit GCC shall characterize a weaker property.

Note that we have:

$$C^-(\infty) > 0 \implies \omega \text{ satisfies GCC} \implies \omega \text{ satisfies aeit GCC}.$$

2.2. The key equivalence relation. Let us turn to the structural-geometric definition, whose goal is to take into account the finer structure of the collision operator. In this subsection, U_1 and U_2 denote two open subsets of $\mathbb{T}^d \times \mathbb{R}^d$.

We first define the binary relation $\mathcal{R}_k$.

Definition 2.6. *We say that $U_1\, \mathcal{R}_k U_2$ if there exist $(x, v_1, v_2) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^d$ with $(x, v_1) \in U_1$, $(x, v_2) \in U_2$ such that $k(x, v_1, v_2) > 0$ or $k(x, v_2, v_1) > 0$.*

We can now define the key equivalence relation on the set of connected components of $\bigcup_{t \geq 0} \phi_{-t}(\omega)$.

Definition 2.7. *Given Ω_1 and Ω_2 two connected components of $\bigcup_{t \geq 0} \phi_{-t}(\omega)$, we say that $\Omega_1 \sim \Omega_2$ if there exist $N \in \mathbb{N}$ and N connected components $(\Omega^{(i)})_{1 \leq i \leq N}$ of $\bigcup_{t \geq 0} \phi_{-t}(\omega)$ such that*

- $\Omega_1\, \mathcal{R}_k\, \Omega^{(1)}$
- *for all $1 \leq i \leq N - 1$, $\Omega^{(i)}\, \mathcal{R}_k\, \Omega^{(i+1)}$,*
- $\Omega^{(N)}\, \mathcal{R}_k\, \Omega_2$.

The relation $\sim$ is an equivalence relation on the set of connected components of $\bigcup_{t\geq 0}\phi_{-t}(\omega)$. In **[11]**, another equivalence relation is defined on the set of connected components of ω, and a natural bijection between the equivalence classes is exhibited, so that all results can actually be stated for both of them. Nevertheless, in proofs, it is sometimes more convenient to choose one or the other.

3. On convergence to equilibrium

We start by defining the weighted L^2 norm adapted to the analysis of the linear Boltzmann equation (1.1), and by recalling the well-posedness in the associated space.

Definition 3.1. *We define the $\mathcal{L}^2$ norm as follows*

$$\|f\|_{\mathcal{L}^2} := \left(\int_{\mathbb{T}^d\times\mathbb{R}^d} |f|^2 \frac{e^{V(x)}}{\mathcal{M}(v)}\, dv\, dx\right)^{1/2}$$

and denote by $\langle\cdot,\cdot\rangle_{\mathcal{L}^2}$ the associated inner product.

***Proposition 3.1* (Well-posedness of the linear Boltzmann equation)**
Assume that $f_0\in\mathcal{L}^2$. Then there exists a unique $f\in C^0(\mathbb{R};\mathcal{L}^2)$ solution of (1.1) satisfying $f|_{t=0}=f_0$, and we have

$$\text{for all } t\geq 0,\quad \frac{d}{dt}\|f(t)\|^2_{\mathcal{L}^2} = -D(f(t)), \tag{3.1}$$

where

$$D(f)=\frac{1}{2}\int_{\mathbb{T}^d} e^V\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\frac{k(x,v',v)}{\mathcal{M}(v)}+\frac{k(x,v,v')}{\mathcal{M}(v')}\right) \times \mathcal{M}(v)\mathcal{M}(v')\left(\frac{f(t,x,v)}{\mathcal{M}(v)}-\frac{f(t,x,v')}{\mathcal{M}(v')}\right)^2 dv'\, dv\, dx$$

is non-negative.

Note that the **dissipation term** $D(f)$ is obtained from $D(f):=-2\langle C(f),f\rangle_{\mathcal{L}^2}$, where

$$C(f)=\int_{\mathbb{R}^d} k(x,v',v)f(v')\,dv' - \left(\int_{\mathbb{R}^d} k(x,v,v')\,dv'\right) f(v)$$

is the collision operator.

As a first result, we have the following chacterization to convergence towards an equilibrium.

***Theorem 3.1* ([11]).** *The following statements are equivalent.*

1. *The set ω satisfies the aeit GCC.*
2. *For all $f_0\in\mathcal{L}^2$, there exists a stationary solution Pf_0 such that*

$$\|f(t)-Pf_0\|_{\mathcal{L}^2}\to_{t\to+\infty} 0,$$

where $f(t)$ is the solution of (1.1) with initial datum f_0.

If (1) or (2) holds, we can actually describe the set of stationary solutions, in terms of the equivalence classes of the equivalence relation $\sim$. As a matter of fact, such a precise description plays a crucial role in our proof.

Of course, among all possible stationary solutions of (1.1), the Maxwellian equilibria particularly stand out. Thus, a natural question is to characterize when the equilibrium which is reached is necessarily a Maxwellian one. This is the purpose of the following theorem (which is actually a particular case of Theorem 3.1 where the description of stationary state Pf_0 is simpler).

Theorem 3.2 **([11]).** *The following statements are equivalent.*

(i.) The set ω satisfies the aeit GCC and there exists one and only one equivalence class for the equivalence relation $\sim$.

(ii.) For all $f_0 \in \mathcal{L}^2$, we have

$$\left\| f(t) - \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} f_0 \, dv \, dx \right) \frac{e^{-V}}{\int_{\mathbb{T}^d} e^{-V} \, dx} \mathcal{M}(v) \right\|_{\mathcal{L}^2} \to_{t \to +\infty} 0.$$

where $f(t)$ is the solution of (1.1) *with initial datum f_0.*

If k satisfies the property that for $x \in p_x(\omega)$ (where $p_x : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{T}^d$ is the canonical projection), the set $p_x^{-1}(\{x\}) \cap \omega$ is contained in one and only one connected component of ω (this is satisfied for instance when ω is connected or $\omega = \omega_x \times \mathbb{R}^d$), then (i.) is equivalent to the following (easier to check) property:

(i'.) The set ω satisfies the aeit GCC and $\bigcup_{t \geq 0} \phi_{-t}(\omega)$ is connected.

Theorems 3.1 and 3.2 hold also for more general phase spaces (up to appropriate adaptations of the geometric definitions, see **[11]**):

- T^*M, where M is a compact Riemannian manifold (without boundary),
- $\Omega \times \mathbb{R}^d$, where Ω is a piecewise C^1 domain of $\mathbb{R}^d$, with specular boundary conditions.

3.1. An example in the case of several equivalence classes. In this section, we describe a situation in which convergence to equilibrium occurs, but the equilibrium ultimately reached Pf_0 is not the projection of the initial datum on the global Maxwellian equilibrium. In other words, in the following example, Theorem 3.1 applies but it turns out that the equivalence relation $\sim$ has two equivalence classes, so that Theorem 3.2 does not apply. This motivates a deeper investigation of the equivalence classes for $\sim$ (and their link with the kernel of the linear Boltzmann operator), that the interested reader may find in **[11]**.

Consider here $(x, v) \in \mathbb{T} \times \mathbb{R}$ and assume that $V = 0$, so that $\phi_t(x, v) = (x + tv, v)$. Take $\varphi_\pm \in C^0 \cap L^\infty(\mathbb{R})$, such that $\varphi_\pm(v) > 0$ for all $v \in \mathbb{R}^\pm_*$ and $\varphi_\pm(v) = 0$ for all $v \in \mathbb{R}^\mp$.

Consider now the collision kernel

$$k(x, v', v) = \mathcal{M}(v) \left[\varphi_+(v)\varphi_+(v') + \varphi_-(v)\varphi_-(v') \right].$$

In this situation, we have

$$\omega := \{\mathbb{T} \times \mathbb{R}^-_*\} \cup \{\mathbb{T} \times \mathbb{R}^+_*\}$$

and hence

$$\bigcup_{t\geq 0} \phi_{-t}(\omega) = \{\mathbb{T} \times \mathbb{R}^-_*\} \cup \{\mathbb{T} \times \mathbb{R}^+_*\}.$$

As a consequence, the aeit GCC is satisfied, but $\sim$ has two equivalence classes.

For the Boltzmann equation associated to this collision kernel k, one can prove that we have convergence to

$$Pf_0 = \left(\int_{\mathbb{T}\times\mathbb{R}^-_*} f_0 \, dvdx\right) \mathcal{M}(v)\mathbb{1}_{v<0} + \left(\int_{\mathbb{T}\times\mathbb{R}^+_*} f_0 \, dvdx\right) \mathcal{M}(v)\mathbb{1}_{v>0}.$$

3.2. The case of free transport in the torus. As a corollary of Theorem 3.2, we prove the following.

Corollary 3.1. *Assume that $V = 0$, and that $\omega = \omega_x \times \mathbb{R}^d$, $\omega_x \neq \emptyset$. Then for all $f_0 \in \mathcal{L}^2$, we have*

$$\left\| f(t) - \left(\int_{\mathbb{T}^d\times\mathbb{R}^d} f_0 \, dv \, dx\right) \mathcal{M}(v) \right\|_{\mathcal{L}^2} \to_{t\to+\infty} 0,$$

where $f(t)$ is the unique solution to (1.1) *with initial datum f_0.*

This is the consequence of the "nice" properties of the geodesic flow on $\mathbb{T}^d \times \mathbb{R}^d$. This situation is not generic. Indeed, for any $\omega_x \neq \mathbb{T}^d$, there exits a an arbitrarily small potential V for which the conclusions of Corollary 3.1 are false (see [**11**]).

Corollary 3.1 is for instance relevant for the variant of (1.2)

$$\partial_t f + v \cdot \nabla_x f = \sigma(x) \left(\left(\int_{\mathbb{R}^d} f \, dv \right) \mathcal{M}(v) - f \right),$$

for $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$, $\sigma \geq 0$, $\sigma \neq 0$.

3.3. Some elements of the proof of Theorem 3.2. Let us now provide some ideas for the proof of Theorem 3.2 (Theorem 3.1 is based on related ideas and requires the description of the set of stationary solutions using the equivalence classes of $\sim$ – this will not be evoked here).

Let us first introduce a **Unique Continuation Property** type property, which turns out to be equivalent to (i.) and (ii.).

Definition 3.2. *We say that the set ω satisfies the Unique Continuation Property (for short UCP) if the only solution $f \in C^0_t(\mathcal{L}^2)$ to*

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f = 0, \\ C(f) = 0, \end{cases} \tag{3.2}$$

is $f = \left(\int_{\mathbb{T}^d\times\mathbb{R}^d} f \, dv \, dx\right) \frac{e^{-V}}{\int_{\mathbb{T}^d} e^{-V} \, dx} \mathcal{M}(v)$.

We prove that (i.) implies UCP by an analysis of the kernel of C and an argument of propagation of information along characteristics. Second, we prove that (ii.) implies (i.) by arguing by contraposition; assuming that (i.) does not hold, we construct examples of solutions to (1.1) which do not converge to a Maxwellian as $t \to +\infty$.

We shall focus on the proof of the fact that UCP implies (ii.). This is based on the fact that the square of the $\mathcal{L}^2$ norm (or **energy**) of a solution $f(t)$ of (1.1) is damped via the explicit dissipation identity (3.1).

A difficulty comes from the fact that we do **not** have the controls

$$D(f) \gtrsim \left\| f - \left(\int f\, dv \right) \mathcal{M}(v) \right\|_{\mathcal{L}^2} \quad \text{or} \quad D(f) \gtrsim \|C(f)\|_{\mathcal{L}^2},$$

Fortunately, a detailed study of D shows that the **weak coercivity** property holds:

$$D(f) = 0 \implies C(f) = 0.$$

(And this holds also in the case of several equivalence classes.)

We argue by contradiction. We take g_0 with zero mean, denying the decay and consider an associated sequence $h_n(t, x, v) := g(t_n + t, x, v)$, $t_n \to +\infty$, such that

$$\|h_n(0)\|_{\mathcal{L}^2} \to \alpha > 0. \tag{3.3}$$

We further impose that $t_{n+1} - t_n \to +\infty$.

By the dissipation identity (3.1), we deduce that

$$\int_0^{t_{n+1}-t_n} D(h_n)\, dt \to 0. \tag{3.4}$$

The core of our analysis is a **Uniqueness-Compactness** argument: we assume, up to a subsequence, that $h_n \rightharpoonup h$ and show that the weak limit h satisfies the linear Boltzmann equation (1.1) and that $C(h) = 0$, using (3.4) and the weak coercivity property. Thus, h satisfies (3.2). Therefore, by the unique continuation property (and the fact that h has zero mean), we conclude that $h = 0$.

Then, we study the sequence of defect measures $\nu_n = |h_n|^2$ and show that the weak limit is $\nu = 0$, which yields a contradiction with (3.3).

Our analysis uses the structure of the linear Boltzmann equation, which is made of a "**kinetic transport + damping**" part and a **relatively compact** part (the term involving the average in velocity of f in the collision operator). That the averaged term is compact is proved via appropriate versions of the **averaging lemmas** of Golse, Lions, Perthame, Sentis [**9**] and DiPerna, Lions, Meyer [**7**].

We also point out that in the study of the sequence of defect measures, there is a possible loss of mass at infinity (in velocity), which is cured by using the **Maximum principle** for the linear Boltzmann equation (and an approximation procedure with initial data in weighted L^∞ spaces).

4. On exponential convergence to equilibrium

Concerning **exponential** convergence to equilibrium, we prove the next result.

Theorem 4.1 **([11]).** *The following two statements are equivalent:*

(a.) $C^-(\infty) > 0$.

(b.) There exists $C > 0, \gamma > 0$ *such that for any* $f_0 \in \mathcal{L}^2$*, the unique solution to* (1.1) *with initial datum* f_0 *satisfies for all* $t \geq 0$*,*

$$\left\| f(t) - \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} f_0 \, dv \, dx \right) \frac{e^{-V}}{\int_{\mathbb{T}^d} e^{-V} \, dx} \mathcal{M}(v) \right\|_{\mathcal{L}^2} \leq C e^{-\gamma t} \left\| f_0 - \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} f_0 \, dv \, dx \right) \frac{e^{-V}}{\int_{\mathbb{T}^d} e^{-V} \, dx} \mathcal{M}(v) \right\|_{\mathcal{L}^2}.$$

The exact analogue of this theorem was previously obtained by Bernard and Salvarani **[3]** for the following linear equation

$$\partial_t f + v \cdot \nabla_x f = \sigma(x) \int_{\mathbb{S}^{d-1}} [k(v', v) f(v') - k(v, v') f(v)] \, dv',$$

for $(t, x, v) \in \mathbb{R}^+ \times \mathbb{T}^d \times \mathbb{S}^{d-1}$, $d \in \mathbb{N}^*$, with $k > 0$, $\sigma \geq 0$. We note that the convergence is in the L^1 norm and the methods involved in **[3]** are different from those presented here.

We also prove an analogue of Theorem 4.1 for Riemannian manifolds. The implication (a.) $\implies$ (b.) holds as well for bounded piecewise C^1 domains Ω with specular reflection, but for a more restrictive class of collision kernels, and with a technical regularity assumption on $\partial\omega \cap \partial\Omega$ (automatically satisfied if $\partial\omega \cap \partial\Omega = \emptyset$).

The problem comes from the fact that averaging lemmas do not yield compactness up to the boundary. To bypass this difficulty, we adapt arguments from Guo **[10]** in order to show that there is no concentration of mass near the boundary $\partial\omega \cap \partial\Omega$ [(1)]. We refer to **[11]** for this point.

The general study of convergence to equilibrium (i.e. Theorem 3.1) allows to prove the following **rigidity** property of the Maxwellian equilibrium with respect to exponential convergence, that we state as a Proposition.

Proposition 4.1 **([11]).** *Assume that there exists* $C > 0, \gamma > 0$ *such that the following holds. For any* $f_0 \in \mathcal{L}^2$*, there is a stationary solution* Pf_0 *such that*

$$\|f(t) - Pf_0\|_{\mathcal{L}^2} \leq C e^{-\gamma t} \|f_0 - Pf_0\|_{\mathcal{L}^2}.$$

Then $C^-(\infty) > 0$*. In particular, we have*

$$Pf_0 = \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} f_0 \, dv \, dx \right) \frac{e^{-V}}{\int_{\mathbb{T}^d} e^{-V} \, dx} \mathcal{M}(v).$$

1. Such a problem of course also occurs for the proof of Theorems 3.1 and 3.2 in bounded domains with specular reflection, but we can appeal to the maximum principle and an equi-integrability argument there.

To conclude this section, let us give some ideas for the proof of Theorem 4.1
• Proof of $(a.) \implies (b.)$. Recalling the dissipation identity (3.1), we use the fact that the exponential decay is equivalent to the following **observability** inequality (relating the dissipation and the energy at time 0): there exist $K, T > 0$ such that for all $f_0 \in \mathcal{L}^2$ with zero mean, we have

$$K \int_0^T D(f(t))\, dt \geq \|f_0\|^2_{\mathcal{L}^2}, \tag{4.1}$$

where f is the solution of (1.1) with initial datum f_0.

The observability inequality is proved *via* a contradiction argument close to that used for Theorem 2. The possible loss of mass at infinity is cured by studying directly the evolution of the $\mathcal{L}^2$ norms, using a Duhamel formula.
• Proof of $(a.) \implies (b.)$. We use a geometric optics type construction, by considering a sequence of initial conditions concentrating on an "undamped trajectory", in order to show that the observability inequality (4.1) does not hold.

5. Localization properties of the spectrum

We turn to a direct study of the spectrum of the linear Boltzmann operator which can be written as

$$A := A_0 + K,$$

where

$$(A_0 f)(x,v) = (v \cdot \nabla_x - \nabla_x V \cdot \nabla_v) f(x,v) + \left(\int k(x,v,v')\, dv' \right) f(x,v),$$
$$(Kf)(x,v) = -\int k(x,v',v) f(x,v')\, dv'.$$

with domain

$$D(A) = D(A_0) = \{f \in \mathcal{L}^2,\ (v \cdot \nabla_x - \nabla_x V \cdot \nabla_v) f \in \mathcal{L}^2\}.$$

We consider in this section the following notions of spectra for A.

Definition 5.1. *We denote by:*
• $\sigma(A)$ *the* **spectrum** *of* A*,*
• $\sigma_p(A)$ *the* **point spectrum** *of* A*, that is, the set of eigenvalues of* A *(i.e* $\lambda \in \mathbb{C}$ *such that* $\mathrm{Ker}(A - \lambda I) \neq \{0\}$*),*
• $\sigma_e(A)$ *the* **essential spectrum** *of* A*, that is, the largest subset of* $\sigma(A)$ *which is stable with respect to (relatively) compact perturbations* (2)*.*

We mention that the whole study of **[12]** is performed in the framework of Riemannian manifolds (without boundary), but we focus on the (flat) torus case for the sake of simplicity.

2. Of course, there exist several finer notions of essential spectrum, which are actually considered in **[12]**, but we stick to this simple definition in this note.

5.1. Rough localization properties. We first prove a theorem which shows how the spectrum of A is localized in the complex plane.

Theorem 5.1 **([12]).** *The following hold:*

1. $\overline{\sigma(A)} = \sigma(A)$.
2. *Let* $L_\infty := \sup_{x,v} \int \left(\frac{k(x,v',v)}{\mathcal{M}(v)} + \frac{k(x,v,v')}{\mathcal{M}(v')} \right) \mathcal{M}(v')\, dv'$ *and*
$$\Sigma_\infty := \{z \in \mathbb{C},\, 0 \le \mathrm{Re}(z) \le L_\infty\}.$$
Then we have $\sigma(A) \subset \Sigma_\infty$.
3. *If* ω *satisfies the aeit GCC, then* $\sigma_p(A) \cap i\mathbb{R} = \{0\}$.
4. $\sigma_e(A) = \sigma_e(A_0) \subset \Sigma := \{z \in \mathbb{C},\, C^-(\infty) \le \mathrm{Re}(z) \le C^+(\infty)\}$.
5. *For* $\varepsilon > 0$, *define*
$$\Sigma_\varepsilon := \left\{z \in \mathbb{C},\, C^-(\infty) - \varepsilon < \mathrm{Re}(z) < C^+(\infty) + \varepsilon\right\}.$$
Then for any $\varepsilon > 0$, *the set* $\sigma(A) \cap \Sigma_\varepsilon^c$ *is made of a finite number of isolated eigenvalues of finite multiplicity.*

Items (1) and (2) are straightforward properties of the spectrum. Item (3) is a consequence of Theorem 3.1. Finally, Items (4) and (5) show that the spectrum of the linear Boltzmann operator is contained in a strip delimited by the Lebeau constants $C^-(\infty)$ and $C^+(\infty)$, plus some discrete eigenvalues which can only accumulate on the edge of the aforementioned strip. Item (5) is reminiscent of classical properties of the linear Boltzmann operator (see for instance [**20, 21**]), the main novelty here follows from the finer bounds of Item (4).

We also obtain the following result.

Theorem 5.2 **([12]).** *Assume that* $C^-(\infty) > 0$. *For any* $\varepsilon > 0$ *such that* $C^-(\infty) - \varepsilon > 0$, *there exists* $n_\varepsilon \in \mathbb{N}$ *and a projector* $\Pi_\varepsilon : \mathcal{L}^2 \to \mathcal{L}^2$ *satisfying* $\Pi_\varepsilon A = A\Pi_\varepsilon$ *and* *rank*$(\Pi_\varepsilon) = n_\varepsilon$ *such that the following holds.*

There exists $C_\varepsilon > 0$, *such that for any* $f_0 \in (I - \Pi_\varepsilon)\mathcal{L}^2$, *the unique solution* $f(t)$ *to* (1.1) *with initial datum* f_0 *satisfies for all* $t \ge 0$, $f(t) \in (I - \Pi_\varepsilon)\mathcal{L}^2$ *and*

$$\begin{aligned}\left\| f(t) - \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} f_0 \, dv dx \right) \frac{e^{-V(x)}}{\int_{\mathbb{T}^d} e^{-V(x)}\, dx} \mathcal{M} \right\|_{\mathcal{L}^2} \\ \le C_\varepsilon e^{-(C^-(\infty) - \varepsilon)t} \left\| f_0 - \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} f_0 \, dv dx \right) \frac{e^{-V(x)}}{\int_{\mathbb{T}^d} e^{-V(x)}\, dx} \mathcal{M} \right\|_{\mathcal{L}^2}.\end{aligned}$$

Theorems 5.1 and 5.2 may be seen as counterparts for the linear Boltzmann Equation to some known spectral properties of the damped wave operator and the damped wave group proved in Lebeau [**16**] and Koch, Tataru [**15**]. In particular, Theorem 5.2 provides another proof of Theorem 4.1 and can be understood as a partial converse of the following result.

Proposition 5.1 **([12]).** *Assume that Theorem 4.1* (*b.*) *holds. Then, we have* $\gamma \le C^-(\infty)$.

This proposition entails that the "best decay rate" is less or equal to $C^-(\infty)$. Loosely speaking, Theorem 5.2 proves that decay at rate $C^-(\infty)$ is almost reached up to a finite dimensional subspace.

5.2. Elements of the essential spectrum. We have seen in Theorem 5.1 that the essential spectrum of the linear Boltzmann equation is contained in a strip delimited by the Lebeau constants. A natural problem is to understand which points of the strip do indeed belong to the essential spectrum.

We show that some values or averages of the damping function

$$b(x,v) := \int_{\mathbb{R}^d} k(x,v,v')\,dv'$$

(and if possible, the associated complex straight line) belong to the essential spectrum.

To this aim, we define (whenever it converges) the Birkhoff average of b along the hamiltonian flow of H by

$$\langle b\rangle_\infty(x,v) = \lim_{T\to+\infty} \frac{1}{T}\int_0^T b\circ\phi_t(x,v)dt.$$

We have the following theorems.

Theorem 5.3 **([12]).** *Suppose that $V=0$ and b is continuous. For all $(x_0,v_0)\in\mathbb{T}^d\times\mathbb{R}^d$ we have, for all $\langle b\rangle_\infty(x_0,v_0)+i\mathbb{R}\subset\sigma_e(A)$.*

Theorem 5.4 **([12]).** *Suppose that $V(x)=V_1(x_1)+\cdots+V_d(x_d)$.*

- *Let $y\in\mathbb{T}^d$ such that $\nabla V(y)=0$, $D^2V(y)=0$, $D^3V(y)=0$. Then we have $b(y,0)+i\mathbb{R}\subset\sigma_e(A)$.*
- *Let $(x,v)\in\mathbb{T}^d\times\mathbb{R}^d\setminus\{0\}$ such that $t\mapsto\phi_t(x,v)$ is periodic (with a nontrivial period). Then we have $\langle b\rangle_\infty(x,v)+i\mathbb{R}\subset\sigma_e(A)$.*

The results we prove are actually more precise, as we show that these averages belong to the so-called Weyl spectrum of A, i.e. $\lambda\in\mathbb{C}$ such that there exist

$$u_n\in D(A) \text{ for all } n\in\mathbb{N},\quad \|u_n\|_{\mathcal{L}^2}=1,\quad u_n\rightharpoonup 0,\quad \|(\lambda I-A)u_n\|_{\mathcal{L}^2}\to 0.$$

These theorems are special cases of general results in the Riemannian framework; similar statements concerning periodic averages hold if the dynamics associated to H is assumed to be **completely integrable** near the periodic trajectory (e.g. the geodesic flow on the sphere $\mathbb{S}^2$).

We also mention that one can obtain some non periodic averages which are well approximated by such periodic averages, see **[12]**. For instance, Theorem 5.3 can be seen as a combination of these two facts.

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