Stability issues in the quasineutral limit of the one-dimensional Vlasov-Poisson equation

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Abstract

This work is concerned with the quasineutral limit of the one-dimensional Vlasov-Poisson equation, for initial data close to stationary homogeneous profiles. Our objective is threefold: first, we provide a proof of the fact that the formal limit does not hold for homogeneous profiles that satisfy the Penrose *instability* criterion. Second, we prove on the other hand that the limit is true for homogeneous profiles that satisfy some monotonicity condition, together with a symmetry condition. We handle the case of well-prepared as well as ill-prepared data. Last, we study a stationary boundary-value problem for the formal limit, the so-called quasineutral Vlasov equation. We show the existence of numerous stationary states, with a lot of freedom in the construction (compared to that of BGK waves for Vlasov-Poisson): this illustrates the degeneracy of the limit equation.

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1 Introduction

The one-dimensional Vlasov-Poisson equation and its quasineutral limit. We study the dynamics of electrons in a plasma, in the presence of ions that are assumed to be immobile and uniformly distributed in space. We assume that this dynamics is described by the Vlasov-Poisson equation. We introduce a positive parameter ε, defined as the ratio of the so-called *Debye length* of the plasma to the typical size of the domain. Loosely speaking, the Debye length is the typical length of electrostatic interaction and in most physical situations, the ratio ε is small, that is $\varepsilon \ll 1$. For a physically oriented discussion on the points mentioned above (and below), we refer to [35].

Throughout this paper, we will focus on the one-dimensional and periodic (in space) case. In this framework, we define $f_\varepsilon(t,x,v)$ the distribution function of the electrons, for $t \in \mathbb{R}^+$, $x \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$ and $v \in \mathbb{R}$, so that $f_\varepsilon(t,x,v)\,dv\,dx$ can be interpreted as the probability of finding electrons at time t with position close to x and velocity close to v. We also introduce the electric potential $V_\varepsilon(t,x)$ and the associated electric field $-\partial_x V_\varepsilon(t,x)$. After nondimensionalization (see [35]), the rescaled 1D Vlasov-Poisson equation reads

$$\begin{cases} \partial_t f_\varepsilon + v\,\partial_x f_\varepsilon - \partial_x V_\varepsilon\,\partial_v f_\varepsilon = 0, \\ -\varepsilon^2\,\partial_x^2 V_\varepsilon = \displaystyle\int f_\varepsilon dv - 1, \end{cases} \tag{1.1}$$

with

an initial condition $f_{0,\varepsilon} \in L^1(\mathbb{T}\times\mathbb{R})$ such that $f_{0,\varepsilon} \geq 0$, $\int f_{0,\varepsilon} dvdx = 1$.

The energy associated to the system (1.1) is the following functional

$$\begin{aligned} \mathcal{E}_\varepsilon[f_\varepsilon] &:= \frac{1}{2}\int f_\varepsilon |v|^2 dvdx + \frac{\varepsilon^2}{2}\int |\partial_x V_\varepsilon[f_\varepsilon]|^2 dx \\ &= \frac{1}{2}\int f_\varepsilon |v|^2 dvdx + \frac{1}{2}\int V_\varepsilon[f_\varepsilon]\,\rho_\varepsilon\,dx. \end{aligned} \tag{1.2}$$

We have used here the notation $V_\varepsilon[f_\varepsilon]$ in order to emphasize that the potential V_ε depends on the distribution function f_ε. We will often forget to mention explicitly this dependance in the sequel, in order to lighten the notations.

In the following, we consider (most of the time) global strong solutions to the system (1.1), with bounded initial energy $\mathcal{E}_\varepsilon[f_{0,\varepsilon}]$. This entails that the energy $\mathcal{E}_\varepsilon(t) = \mathcal{E}_\varepsilon[f_\varepsilon(t)]$ is preserved, as so is $\int Q(f_\varepsilon)\,dxdv$, for any continuous function Q (and for any ε). Remark that in this one-dimensional case, the strongness assumption is not a huge restriction since there is a weak-strong stability principle for solutions whose density ρ_ε remains bounded in L^∞ for all times [37, Theorem 1.9]. This last property can be ensured for instance if $f_{0,\varepsilon}$ is bounded from above by a profile $g_0(|v|)$ that is decreasing, bounded, and integrable.

In this paper, we shall study the behavior of solutions to (1.1) as $\varepsilon \to 0$, a limit that we shall refer to as the *quasineutral limit*.

The formal limit : the quasineutral Vlasov equation. We begin with a brief formal analysis of the limit $\varepsilon \to 0$. Let us assume that in some sense, we have $f_\varepsilon \to f$ and $V_\varepsilon \to V$. The formal limit is straightforward : only the Poisson equation is affected and degenerates into $\rho := \int f\, dv = 1$. The limit system then reads

$$\begin{cases} \partial_t f + v\, \partial_x f - \partial_x V\, \partial_v f = 0, \\ \int f dv = 1, \end{cases} \tag{1.3}$$

which we shall refer to as the *quasineutral Vlasov equation*.

We observe that the total energy associated to this system corresponds only to the kinetic part of (1.2)

$$\mathcal{E}[f] = \frac{1}{2} \int f\, |v|^2 dv dx. \tag{1.4}$$

The unknown potential V can be seen as a Lagrange multiplier, or a pressure, associated to the incompressibility constraint $\rho = 1$. But an explicit equation for V is hidden in the equation. Indeed, if we integrate the transport equation (1.3) with respect to v, we get the “zero divergence” constraint on the current j

$$\partial_x j = 0, \qquad \text{where } \ j(t,x) := \int v f(t,x,v) dv, \tag{1.5}$$

which implies that j is only a function of time. Next, we use the equation on the local momentum, obtained after multiplication of equation (1.3) by v and integration in v

$$\partial_t j + \partial_x \left(\int v^2 f dv + V \right) = 0. \tag{1.6}$$

It implies that $\partial_t j$ which is only a function of time, is also a gradient in x. The only possibility is that $\partial_t j = \partial_x \left(\int v^2 f dv + V \right) = 0$, so that j should be constant in time and position (and *a fortiori* at time 0).

In particular, we get a kind of “pressure law” : the potential V is, up to a constant, the opposite of the local kinetic energy

$$V = - \int v^2 f dv. \tag{1.7}$$

For this reason this quasineutral limit can somehow be seen as a kinetic version of the classical incompressible limit of fluid mechanics (we refer to Gallagher [23] for a review on this topic). To go even further into the analogy, System (1.3) and its higher dimensional dimension generalizations can be interpreted as the kinetic version of the incompressible Euler system, as it has been pointed out by Brenier [14]. One interesting feature of this system is that it still makes sense in one dimension, which is of course not the case for the incompressible Euler equation. For a numerical analysis of the pressure law (1.7), we refer to [21], where it is used in an attempt to get an asymptotic preserving scheme in the quasineutral limit.

To the best of our knowledge, only little is known about the (local) well-posedness of the quasineutral Vlasov equation (1.3). The local in time existence of *analytic* solutions is shown in [13, Theorem 1.1]; the case under consideration here corresponds to $\beta = \sigma = \alpha = 0$ in this reference. Similar results [35, 39, 6] have also been proved for a related system of equations, namely

$$\begin{cases} \partial_t f + v\,\partial_x f - \partial_x V \partial_v f = 0, \\ V = \rho - 1, \end{cases} \tag{1.8}$$

which was called *Vlasov-Dirac-Benney* by Bardos [4]. The local in time existence in Sobolev spaces of monotonic solutions to (1.8) (precisely, solutions that are for any x, increasing and then decreasing in v) is shown in [12, 4]. We also refer to the very recent work [5]. However, at least to our knowledge, such a result is not known for the quasineutral Vlasov equation (1.3). Remark also that [4] contains an interesting argument, that suggests ill-posedness for non monotone (in the above sense) and non analytic initial conditions.

The problem of well-posedness of (1.3) is not the main topic of this paper: we rather focus on the justification of the quasineutral limit (although the two questions are of course related). Does the asymptotics

$$f_\varepsilon(t, x, v) \approx f(t, x, v), \tag{1.9}$$

where f_ε satisfies (1.1) and f satisfies (1.3), holds when $\varepsilon \to 0$ and $f_{0,\varepsilon} \approx f_{|t=0}$? In this paper, we will restrict ourselves to the particular case where f is an homogeneous equilibrium: $f(t, x, v) = \mu(v)$. In other words, we are in particular interested in the *the question of stability (or instability) around homogeneous equilibria in the quasineutral limit*. The general case seems much more difficult to handle, but the study of the quasineutral limit around such equilibria already gives an overview of the problems raised by this limit.

Unstable and stable homogeneous equilibria for Vlasov-Poisson. Before going on, we shall now recall some well-known facts about the stability or instability of homogeneous equilibria (homogeneous meaning that the profile only depends on v) for the classical Vlasov-Poisson equation

$$\begin{cases} \partial_t f + v\,\partial_x f - \partial_x V\,\partial_v f = 0, \\ -\,\partial_x^2 V = \displaystyle\int f dv - 1. \end{cases} \tag{1.10}$$

The question of the linear stability of such profiles is now quite well understood, and some important results about the non-linear stability were also proved in the last years, culminating in the proof of nonlinear Landau damping by Mouhot and Villani [47] (see also the very recent paper by Bedrossian, Masmoudi and Mouhot [8] and the non-damping results of Lin and Zheng [40, 41]). In the sixties, O. Penrose gave in [49] a famous criterion for the existence of unstable modes (or generalized eigenvalues) for the linearized Vlasov-Poisson equation around an homogeneous profile $\mu(v)$. It is related to the existence of instabilities of kinetic nature, often referred to as *two stream instabilities* which appear for homogeneous distributions in v, with two or more maxima. In dimension one, it may be stated in the following way.

Definition 1.1. *We say that an homogeneous profile $\mu(v)$, such that $\int \mu\, dv = 1$, satisfies the Penrose instability criterion if μ has a local minimum point $\bar{v}$ such that the following inequality holds*

$$\int_{\mathbb{R}} \frac{\mu(v) - \mu(\bar{v})}{(v - \bar{v})^2}\, dv > 0. \tag{1.11}$$

If the local minimum is flat, i.e. is reached on an interval $[\bar{v}_1, \bar{v}_2]$, then (1.11) has to be satisfied for all $\bar{v} \in [\bar{v}_1, \bar{v}_2]$.

Taking the regularity and decrease at infinity aside, this criterion is a necessary and sufficient condition for the existence of unstable modes in the linearized equation around the homogeneous profile μ when the Vlasov-Poisson equation (1.10) is posed in the whole space $\mathbb{R}$ (for the position variable x).

When we are restricted to a torus $\mathbb{T}_M := \mathbb{R}/(M\mathbb{Z})$ of size $M > 0$, then it becomes more involved to give a necessary and sufficient condition. But there still exists a rather straightforward necessary criterion: the linearized VP equation (1.10) around μ is unstable only if μ has a local minimum $\bar{v}$ such that

$$\int_{\mathbb{R}} \frac{\mu(v) - \mu(\bar{v})}{(v - \bar{v})^2} \, dv > \frac{4\pi^2}{M^2}. \tag{1.12}$$

It is interesting to remark that the above necessary condition becomes actually sufficient if we restrict to profiles that are symmetric around $\bar{v}$, i.e. $\mu(2\bar{v} - v) = \mu(v)$ for all $v \in \mathbb{R}$: see for instance [33, Lemma 2.1] for the case $\bar{v} = 0$, $M = 2\pi$. In fact, for these particular symmetric profiles (and under some smoothness assumptions), Guo and Strauss also gave a nonlinear instability result in [33].

We shall see later that the right criterion for our purposes (quasineutral limit in the torus $\mathbb{T}$) turns out to be (1.11) and we shall give more details on this fact in the section 3.1.

On the contrary, the Penrose criterion suggests that when μ has no local minimum, i.e. μ is increasing and then decreasing, then the profile may be stable (see also [47, Section 2.2]). In fact, it was proved by Marchioro and Pulvirenti in dimension one and two [43], and by Batt and Rein in dimension three [7] that radially decreasing profiles are indeed non linearly stable. Their proof relies on rearrangement inequalities.

To our knowledge, the only result that proves nonlinear stability without any symmetry assumption is the work of Mouhot and Villani on Landau damping: [47, Theorem 2.2] implies some orbital stability, but only for Gevrey perturbations of a Gevrey homogeneous profile.

The mathematical difficulties of the quasineutral limit. The mathematical analysis is more subtle than what the formal analysis would suggest. As already mentioned, we focus on the quasineutral limit around an unstable homogeneous profile $\mu(v)$, i.e. for initial data converging in some sense to $\mu(v)$.

The role of the stable or unstable nature of μ in the analysis of the limit was pointed out by Grenier in [29]. Indeed, a major obstruction to the asymptotics arises when a profile μ satisfies the Penrose instability criterion of Definition 1.1. One can remark that such a profile μ is a stationary solution of (1.1) and also (1.3), with an electric field identically equal to zero, as it is the case for any distribution function depending only on v. In the short proceeding note [29], Grenier explains, without giving a proof, why for such profiles, the formal convergence to the expected system (1.3) is in general false. Basically, the idea is that thanks to some scaling invariance of the Vlasov-Poisson equation (1.1), the instabilities (whenever they exist) develop on a very short timescale of order ε.

A consequence is that the linearized quasineutral or Vlasov-Dirac-Benney equations around an unstable equilibrium have a unbounded spectrum : it possesses eigenvalues with arbitrary high real part. See the analysis in [4, 6] for the Vlasov-Dirac-Benney case.

Even when we consider stable homogeneous profiles (with one and only one "hump"), a second difficulty arises : it is due to the presence of time oscillations of frequency $\mathcal{O}(\varepsilon^{-1})$ and amplitude $\mathcal{O}(\varepsilon^{-1})$ of the electric field, usually referred to as *plasma oscillations* or *Langmuir waves* (see for instance [28]). We will describe these more carefully in Section 4.3. Let us just emphasize here that these oscillations are not damped: they do carry a constant amount of energy, and the problem is very different from an initial boundary layer problem. Therefore, these oscillations have an impact on the formal limit.

State of the art. One of the first mathematical works on the quasineutral limit of the Vlasov-Poisson equation was performed by Brenier and Grenier in [18, 27], using the defect measure approach, originally introduced in [22] for the Euler equation. In these works, the limit of the two first moments in v is studied. In the limit equations, in addition to the terms one could formally guess, two defect measures appear, which account for the lack of compactness, together with time oscillations. Loosely speaking, it is explained that the defect measures are more or less related to the possible very fast instabilities, while the time oscillations are due to the fast Langmuir waves.

In a subsequent work [28], Grenier gives another description of what happens in the quasineutral limit, using a somewhat unusual point of view. He describes the plasma as a superposition of a (possibly uncountable) collection of fluids : the distribution function is written under the form

$$f_\varepsilon(t,x,v) = \int_M \rho_\theta^\varepsilon(t,x)\delta_{v_\theta^\varepsilon(t,x)}(v)\mu(d\theta),$$

where θ is a parameter belonging to some probability space M and μ is a probability measure on that space. This is quite general since any reasonable distribution f may be written under this form (actually in a very large number of ways), and in particular it applies to the "cold electrons" case, i.e. when the sequence of initial distribution $f_{0,\varepsilon}$ converges towards a monokinetic profile of the form

$$f_0(x,v) = \rho_0(x)\delta_{v_0(x)}(v), \tag{1.13}$$

where δ denotes as usual the Dirac measure. Under some strong uniform regularity assumption on the whole sequence of solutions, which ensures that the instability phenomena discussed before (such as two stream instabilities) are not present, he proved that if the fast and undamped plasma oscillations are "filtered", the collection $(\rho_\theta^\varepsilon, v_\theta^\varepsilon)_{\theta\in M}$ converges (up to some extraction) to a solution of a multiphase incompressible Euler system. Moreover, he also shows that the strong regularity assumptions are fulfilled if the sequence of initial conditions converges in some space of analytic functions.

The previous convergence result of Grenier was improved in two directions by Brenier [16] and Masmoudi [44], in the "cold electrons" case only. Brenier introduced the so-called modulated energy method (also referred to as the relative entropy method) and proved stability estimates which entail the convergence for well-prepared initial data towards a dissipative solution of the Euler equation, a very weak notion of solutions introduced by Lions in [42] which satisfy a weak-strong uniqueness principle. In particular, Brenier's technique has the advantage to require weak regularity assumptions. These well-prepared initial conditions correspond exactly to those for which the Langmuir waves vanish in the limit.

The stability estimates of Brenier are natural since monokinetic initial data of the form (1.13) correspond to an extremal case of symmetric and monotonic profiles. He also explained how the results obtained in [18, 27] on the existence of defect measures may lead to the same result. Later, Masmoudi extended in [44] the convergence to non necessarily well-prepared initial data, but with stronger regularity assumptions on the limit equation. His work combines filtration techniques and the modulated energy method.

2 Main results

In this work, we provide three types of results, related to the quasineutral limit and stationary states.

- Let $\mu(v)$ be a profile satisfying the Penrose instability criterion of Definition 1.1, and a technical condition (that essentially forbids the presence of isolated zeros and fast oscillations on its tail, see Definitions 2.1 and 3.1). Our first result asserts that in general, even if $f_{0,\varepsilon} \to \mu$ in $W^{s,1}_{x,v}$ for some $s \in \mathbb{N}$, as $\varepsilon \to 0$, the asymptotics

$$f_\varepsilon(t, \cdot) \rightharpoonup \mu(\cdot), \quad \text{as } \varepsilon \to 0 \tag{2.1}$$

  where f_ε is the solution to (1.1) with initial datum $f_{0,\varepsilon}$, is only true for $t = 0$ and does not hold on an interval of time $[0, T]$, for any $T > 0$, and any $W^{-r,1}_{x,v}$-norm, $r \in \mathbb{N}$. Actually, the results we prove are more accurate, see Theorem 2.1. In other words, we provide a complete proof of the result suggested by Grenier in his note [29] (actually for a larger class of homogeneous equilibria and in general topologies).

- Conversely, we are able to justify the asymptotics (2.1), on any interval of time $[0, T]$, as soon as $\mu(v)$ satisfy a monotonicity and a symmetry condition (along with some other minor technical conditions), see Theorems 2.2 and 2.3. In some sense, it extends in dimension one the stability results of Brenier [16] and Masmoudi [44] to some cases of initial data which do not converge to a monokinetic profile.

- We show the existence of an uncountably infinite number of stationary solutions (or "BGK waves") to a boundary value problem associated to (1.3), with a lot of freedom in the construction. This is due to the possible presence of trapped particles, whose density is shown to depend in a very simple way on the boundary conditions, see Theorem 2.4. We shall discuss the potential consequences on the stability properties of (1.3) below.

Let us now describe more precisely each of these results.

2.1 Unstable case.

Before stating our instability result, we need to introduce a technical condition:

Definition 2.1. *We say that a positive and C^1 profile $\mu(v)$ satisfies the δ-condition if*

$$\sup_{v \in \mathbb{R}} \frac{|\mu'(v)|}{(1 + |v|)\mu(v)} < +\infty. \tag{2.2}$$

In the sequel, we shall also introduce a more general (but also more technical) condition that allows to handle some non-negative profile, see the δ'-condition of Definition 3.1 for details.

We are now in position to state the instability result. As usual, the notation $W^{s,1}_{x,v}$ refers to the classical Sobolev spaces built on $L^1_{x,v}(\mathbb{T} \times \mathbb{R})$ and of order s. In what follows, we say that a profile $\mu(v)$ is smooth if it belongs to $W^{s,1}_v$ for all $s \in \mathbb{N}$. We shall use the following convention in the statement of the theorem: the notation (f_ε) (with the continuous parameter $\varepsilon > 0$) refers in fact to a sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ going to 0 and a sequence $(f_{\varepsilon_k})_{k \in \mathbb{N}}$.

Theorem 2.1. *Let $\mu(v)$ be a smooth profile satisfying the Penrose instability criterion of Definition 1.1. Assume either that μ is positive and satifies the δ-condition of Definition 2.1, or* *that μ is non-negative and satisfies the δ'-condition of Definition 3.1. For any $N > 0$ and $s > 0$, there exists a sequence of non-negative initial data $(f_{0,\varepsilon})$ such that*

$$\|f_{\varepsilon,0} - \mu\|_{W^{s,1}_{x,v}} \leq \varepsilon^N,$$

and denoting by (f_ε) the sequence of solutions to (1.1) with initial data $(f_{0,\varepsilon})$, the following holds:

i) L^1 **instability for the macroscopic observables:** *the density $\rho_\varepsilon := \int f_\varepsilon \, dv$, and the electric field $E_\varepsilon = -\partial_x V_\varepsilon$. For all $\alpha \in [0,1)$, we have*

$$\liminf_{\varepsilon \to 0} \sup_{t \in [0,\varepsilon^\alpha]} \|\rho_\varepsilon(t) - 1\|_{L^1_x} > 0, \qquad \liminf_{\varepsilon \to 0} \sup_{t \in [0,\varepsilon^\alpha]} \varepsilon \, \|E_\varepsilon\|_{L^1_x} > 0. \tag{2.3}$$

ii) **Full instability for the distribution function:** *for any $r \in \mathbb{Z}$, we have*

$$\liminf_{\varepsilon \to 0} \sup_{t \in [0,\varepsilon^\alpha]} \|f_\varepsilon(t) - \mu\|_{W^{r,1}_{x,v}} > 0. \tag{2.4}$$

For $t = 0$, by construction, we have $\lim_{\varepsilon \to 0} \|\rho_{0,\varepsilon} - 1\|_{L^1_x} = \lim_{\varepsilon \to 0} \|E_{0,\varepsilon}\|_{L^1_x} = 0$. This theorem can thus be rephrased as follows: there exist small smooth perturbations of f_0 for which the corresponding solutions of (1.1) do not converge to the expected stationary solution μ in a weak $W^{-r,1}$-sense for any $r \in \mathbb{N}$.

Remark 2.1. *It is possible to lower down the required regularity on the profile μ, see Remark 3.1.*

Remark 2.2. *Keeping the notations of the theorem, we can actually show that for any $r \in \mathbb{N}$,*

$$\liminf_{\varepsilon \to 0} \sup_{t \in [0,\varepsilon^\alpha]} \frac{1}{\varepsilon^r} \|\rho_\varepsilon(t) - 1\|_{W^{-r,1}_x} > 0, \tag{2.5}$$

Note however that the instability is not directly seen, without the weight in ε, in the $W^{-1,1}$-norm of the density. This is due to the preservation of the total energy that provides stability in weak norms on macroscopic observables:

$$\big\|\rho_\varepsilon(t) - 1\big\|_{W^{-1,1}_x} \leq C\varepsilon^2 \|E_\varepsilon\|_1 \leq C\varepsilon \sqrt{\mathcal{E}_\varepsilon[f_{0,\varepsilon}]}.$$

The δ-condition of Definition 2.1 and the δ'-condition of Definition 3.1 are precisely introduced in order to enforce that the sequence of initial data we build in Theorem 2.1 is non-negative, and thus to ensure their physical relevancy (and this is their only purpose). We refer to the paper of Guo and Strauss [34] where the same problem is faced.

The δ-condition (2.2) is quite general: it is satisfied by positive profiles that, for large velocities, do not oscillate too much and decrease slower than some Maxwellian distribution. For instance profiles

- which coincide with a power law for large velocities, i.e.

$$\mu(v) = \frac{\lambda_1}{|v|^{\lambda_2}}, \quad \text{for } v \text{ large enough, with } \lambda_1, \lambda_2 > 0;$$

- which coincide with Maxwellian profiles for large velocities, i.e.

$$\mu(v) = \lambda_1 e^{-\lambda_2 |v|^2}, \quad \text{for } v \text{ large enough, with } \lambda_1, \lambda_2 > 0.$$

Nevertheless, profiles that vanish at some point never satisfy the δ-condition. However the δ'-condition of Definition 3.1 allows to handle profiles that vanish (at infinite order only) or decrease faster than exponentially. As it is not very explicit, we postpone its precise statement to the subsection 3.4 and detail some sufficient conditions in the following Proposition.

Proposition 2.1. *The δ'-condition of Definition 3.1 is satisfied if one of the following holds true:*

i. the profile μ is positive and satisfy for $|v|$ large enough with $\alpha > 1$ and $C_\alpha > 0$

$$\frac{1}{C_\alpha} |v|^\alpha \leq \frac{\mu'(v)}{\mu(v)} \leq C_\alpha |v|^\alpha; \tag{2.6}$$

ii. the profile μ is C^∞ on the whole line $\mathbb{R}$, positive on the union of a finite number of disjoints open interval (a_i, b_i) and equal to zero outside, and for all i, there exists ε_1 such that on $(a_i, a_i + \varepsilon_i)$ (resp. on $(b_i - \varepsilon_i, b_i)$), μ satisfies for some $C_i > 0$ and $\beta_i > 1$ (resp. $C'_i > 0$ and $\beta'_i > 1$)

$$\frac{1}{C_i} |v - a_i|^{-\beta_i} \leq \frac{\mu'(v)}{\mu(v)} \leq C_i |v - a_i|^{-\beta_i}, \tag{2.7}$$

$$\text{resp.} \quad \frac{1}{C'_i} |v - b_i|^{-\beta'_i} \leq \frac{\mu'(v)}{\mu(v)} \leq C'_i |v - b_i|^{-\beta'_i}.$$

The case where some b_i is equal to some a_j is allowed, but only with the additional assumption that the profil μ should vanish at any order (all derivative should vanish) at this point.

A mix of points i. and ii. is also allowed. On the other hand, the δ'-condition of Definition 3.1 is not satisfied if μ has a zero with finite order.

Proposition 2.1 will be proved after Theorem 2.1. It covers a large variety of profiles. Loosely speaking, we have:

- Point *i.* is satisfied by positive profiles that do not oscillate for large velocities and decrease very fast: for instance, $\mu(v) \sim_{|v| \to +\infty} \exp(-|v|^\alpha)$, with $\alpha > 2$.
- Point *ii.* is satisfied by some smooth profiles with compact support: for instance $\mu(v) = \exp\bigl(-(b-v)^{-1}(v-a)^{-1}\bigr)$ on (a, b) and 0 outside.

However, while it is possible to handle zero of infinite order in μ, zeros of finite (and especially small) order may raise difficulties. As a matter of fact, our proof is not relevant if μ possesses a zero with small order. We are not able to say if this is only a technical point, or a more important physical point.

2.2 Stable case.

We now restrict to particular homogeneous stable equilibria. The precise conditions we require are listed in the following definition.

Definition 2.2 (S-stability)**.** *We will say that a profile μ satisfying $\int_{\mathbb{R}} \mu(v)\, dv = 1$ is S-stable if the four following conditions are fulfilled:*

i) Continuity: *μ is continuous on $\mathbb{R}$.*

ii) Finite energy: *$\int \mu(v) v^2 \, dv < +\infty$.*

iii) Monotonocity: *There is $\bar v \in \mathbb{R}$, such that $v \mapsto \mu(v)$ is increasing for $v < \bar v$ and decreasing for $v > \bar v$, and so μ reaches its unique maximum at $\bar v$.*

iv) Symmetry: *For all $v \in \mathbb{R}$, $\mu(2\bar v - v) = \mu(v)$.*

In this case, there is a unique continuous and increasing function $\varphi : (-\infty, 0] \to \mathbb{R}^+$ such that

$$\mu(v) = \varphi\Big(-\frac{|v-\bar v|^2}{2}\Big), \quad \textit{and} \quad \int_{-\infty}^{0} \varphi(u)\sqrt{-u}\, du < +\infty. \tag{2.8}$$

In Theorems 2.2 and 2.3, we will justify the quasineutral limit around such stable equilibria. The stability of the solution f_ε of (1.1) around μ will be controlled thanks to the so-called “Casimir functionals” defined in the following

Definition 2.3. *For any S-stable profile, to which we associate the function φ defined in* (2.8)*, we introduce a function $Q : \mathbb{R}^+ \to \mathbb{R}$ satisfying $Q(0) = 0$ and $Q' = \varphi^{-1}$ on the range of φ. Outside this range, the only condition is that Q' is increasing and continuous so that Q is globally convex and C^1 on $\mathbb{R}^+$. For such a function Q, we can define the associated Casimir functional*

$$H_Q(f) := \int \big(Q(f) - Q(\mu) - Q'(\mu)(f-\mu)\big)\, dvdx, \tag{2.9}$$

which is well defined with value in $[0, +\infty]$.

Remark that there is some freedom in the choice of Q, whose values are imposed only on the range of φ. But all the results will shall give in the sequel are valid independently of the particular choice made for Q.

This “Casimir functional” is a kind of relative entropy for the Vlasov-Poisson equation; it is built in order to be minimized by μ. Similar quantities were originally introduced by Arnold [1, 2] for fluid models. Later, their use was generalized to plasma models in [38]. The first fully rigorous application of these functionals to plasmas was performed by Rein in [50]: he used them to prove the L^2 stability around compactly supported equilibria that are decreasing function of the energy. We also refer to that article for a very clear explanation of their interest. This result was later extended in [19] to non compact equilibria. As we shall see in Section 4, the above quantity (2.9), may control L^p norms

$$\|f_\varepsilon - \mu\|_p^2 \leq \frac{1}{C} H_Q(f_\varepsilon), \qquad \text{for some } p, \qquad \text{for instance } p = 1, 2.$$

We refer to Proposition 4.1 and more generally to section 4.1 for more details. Therefore, all the following results, showing that $H_Q(f_\varepsilon)$ (or H_Q applied to some filtered distribution function) remains small under some assumption on the initial conditions, may be translated in results of stability in more usual norms.

Our first result of stability is given in the following

Theorem 2.2. *Let μ be a S-stable stationary solution to* (1.3) *of the form given in* (2.8)*. Assume that there exists $\eta > 0$, such that μ satisfies*

$$\int \mu(v)(1 + v^{2+\eta})\, dv < +\infty. \tag{2.10}$$

For all $\varepsilon > 0$, let $(f_\varepsilon, V_\varepsilon)$ be a strong solution to (1.1)*, with initial datum $f_{0,\varepsilon}$ and define the “modulated energy”*

$$\mathcal{L}_\varepsilon[f_\varepsilon] := H_Q(f_\varepsilon) + \frac{\varepsilon^2}{2} \int (\partial_x V_\varepsilon)^2 dx. \tag{2.11}$$

Then, $\mathcal{L}_\varepsilon$ is a Lyapunov functional in the sense that

$$\forall t \in \mathbb{R}^+, \qquad \mathcal{L}_\varepsilon[f_\varepsilon(t)] = \mathcal{L}_\varepsilon[f_{0,\varepsilon}].$$

The proof consists in a reformulation of the functional $\mathcal{L}_\varepsilon$ that shows that it is composed only of invariant quantities. Remark that the assumption that μ is S-stable is crucial since otherwise, we can not even define an associated Casimir functional.

Remark 2.3. *It is also possible in this theorem to consider a sequence of global weak solutions to* (1.1) *in the sense of Arsenev [3], in which case the conclusion becomes*

$$\forall t \in \mathbb{R}^+, \qquad \mathcal{L}_\varepsilon[f_\varepsilon(t)] \leq \mathcal{L}_\varepsilon[f_{0,\varepsilon}].$$

The above theorem is particularly useful when the initial potential energy vanishes in the limit, that is when $\|\partial_x V_{0,\varepsilon}\|_2 = o\big(\varepsilon^{-1}\big)$, or in other words when $\|\rho_{0,\varepsilon} - 1\|_{H^{-1}} = o(\varepsilon)$. It corresponds to what we can call well-prepared initial data, for which there are no plasma oscillations in the limit. In this situation, we can express the conclusion of the previous theorem as follows.

Corollary 2.1. *If the sequence of initial data is well-prepared in the sense that $\mathcal{L}_\varepsilon[f_{\varepsilon,0}] \to 0$, then for all $t \geq 0$, $\mathcal{L}_\varepsilon[f_\varepsilon(t)] \to 0$. Moreover the rate of convergence to* 0 *for any positive time is the same as the one at initial time.*

This corollary is thus orthogonal to Theorem 2.1, since it tells us that when the profile is S-stable, then it is not possible to find initial conditions satisfying the conclusions of that theorem.

On the other hand, this does not say much if the data are not well-prepared. For instance, in the case where $V_{0,\varepsilon}$ displays some oscillations in space, then we do not expect the plasma oscillations to vanish, and we have to filter them in order to prove a convergence result. The precise result is the following.

Theorem 2.3. *Let μ be a S-stable stationary solution to* (1.3) *of the form given in* (2.8)*. Assume that there exists $\eta > 0$, such that μ satisfies*

$$\int \mu(v)(1 + v^{2+\eta})\, dv < +\infty. \tag{2.12}$$

For all $\varepsilon > 0$, let $(f_\varepsilon, V_\varepsilon)$ be a global strong solution to (1.1)*, with initial datum $f_{0,\varepsilon}$. For any smooth potential V_0 such that $\partial_{xxx} V_0 \in L^\infty$, we define an associated "modulated free energy"*

$$\begin{aligned} \mathcal{L}^O_\varepsilon(t) := H_Q \left[f_\varepsilon \left(t, x, v - \partial_x V_0(x - \bar{v}t) \sin \frac{t}{\varepsilon} \right) \right] \\ + \frac{1}{2} \int \left[\varepsilon \partial_x V_\varepsilon - \partial_x V_0(x - \bar{v}t) \cos \frac{t}{\varepsilon} \right]^2 dx. \end{aligned} \tag{2.13}$$

Then, we can control the growth of $\mathcal{L}^O_\varepsilon$ in the sense that there exists a constant $K > 0$, depending on $\|\partial_{xx} V_0\|_\infty$ and $\|\partial_{xxx} V_0\|_\infty$, such that for any $t > 0$

$$\forall t \geq 0, \quad \mathcal{L}^O_\varepsilon(t) \leq e^{2\|\partial_{xx} V_0\|_{L^\infty} t} \Big[\mathcal{L}^O_\varepsilon(0) + K\varepsilon\big(1 + \mathcal{E}_{\varepsilon,0} + \mathcal{Q}_{\varepsilon,0}\big) \Big], \tag{2.14}$$

where

$$\mathcal{E}_{\varepsilon,0} := \mathcal{E}_\varepsilon(f_{0,\varepsilon}), \quad \textit{and} \qquad \mathcal{Q}_{\varepsilon,0} := \int \left(|Q|(f_{\varepsilon,0}) + \frac{Q^2(f_{\varepsilon,0})}{f_{\varepsilon,0}} \right) dv dx. \tag{2.15}$$

Of course, this theorem implies the following stability result

Corollary 2.2. *Assume in addition to the hypotheses of Theorem 2.3 that $f_{\varepsilon,0}$ satisfies the (very weak) bound*

$$\mathcal{E}_{\varepsilon,0} + \mathcal{Q}_{\varepsilon,0} = o(\varepsilon^{-1}).$$

If $\mathcal{L}^O_\varepsilon[f_{\varepsilon,0}] \to 0$, then for all $t \geq 0$, $\mathcal{L}^O_\varepsilon[f_\varepsilon(t)] \to 0$.

Note that the best possible rate of convergence is given by ε, contrary to the well-prepared case, and that this best rate is reached when $\mathcal{E}_{\varepsilon,0}$ and $\mathcal{Q}_{\varepsilon,0}$ are uniformly bounded in ε.

Remark 2.4. *Observe that the plasma oscillations which have to be filtered are mostly oscillations in velocity. In fact, the density ρ_ε will remain close to 1. This may be quantified in some cases, for instance when μ is a Maxwellian equilibrium, in which case the Casimir functional is the "usual" relative entropy (see iii) in Proposition 4.1).*

Remark 2.5. *In the ill-prepared case, to obtain stability, we have to add the assumption that $Q_{\varepsilon,0}$ is (not necessarily uniformly) bounded. This is not a huge requirement: for instance, when μ is a Maxwellian distribution, it only requires $f_{\varepsilon,0}(\ln f_{\varepsilon,0})^2$ to be integrable (see again iii) in Proposition 4.1).*

In the case where V_0 does not have a bounded third derivative, an interpolation argument still allows to get the following stability result.

Corollary 2.3. *With the same notations and hypotheses of Theorem 2.3, except that we only assume that $\partial_{xx} V_0 \in L^\infty$, we obtain*

$$\forall t \geq 0, \quad \mathcal{L}^O_\varepsilon(t) \leq 4\, e^{2\|\partial_{xx} V_0\|_{L^\infty} t} \Big[\mathcal{L}^O_\varepsilon(0) + K' \varepsilon^{\frac{2}{3}} \big(1 + \mathcal{E}_{\varepsilon,0} + \mathcal{Q}_{\varepsilon,0}\big)^{\frac{2}{3}} \Big], \tag{2.16}$$

for some constant K' depending only on $\|\partial_{xx} V_0\|_\infty$.

The argument is detailed at the end of Section 4.4. In this case, the best rate of convergence is therefore given by $\varepsilon^{2/3}$.

Remark 2.6. *Alternatively to the Modulated energy (or Casimir functional) technique developed here, the technique of [43, 7] based on rearrangement inequalities may be used to prove the stability in the well-prepared case. But this seems more difficult for the ill-prepared case. Indeed, it relies on the fact that a S-stable profile (for $\bar{v} = 0$), will be the minimizer of the total energy in some class of equi-measurable functions. It does not seem clear how to adapt this argument in the ill-prepared case, mostly because the plasma oscillations do carry some energy in the limit.*

2.3 Locally symmetric solutions to (1.3) are homogeneous.

The two previous theorems of stability rely on a natural monotonicity condition, but also on a symmetry condition on f. Indeed, we use in a crucial way a Casimir functional, that can be constructed only for symmetric profiles.

It is maybe also natural to expect that such conditions could be helpful to prove the convergence for solutions to (1.3) which are not necessarily stationary, but which satisfy the monotonicity condition and the symmetry condition at any time t and any position x

$$\forall\, t \in \mathbb{R}^+,\ \forall\, x \in \mathbb{T},\ \exists\, \bar{v}(t,x),\ \varphi_{t,x} \text{ s.t. } \quad f(t,x,v) := \varphi_{t,x}\Big(-\frac{|v - \bar{v}(t,x)|^2}{2}\Big) \tag{2.17}$$

where $\varphi_{t,x} : \mathbb{R}^- \to \mathbb{R}^+$ is increasing. Then, for such functions, it is easy to build (time and position dependent) relevant Casimir functionals as in Definition 2.3. Unfortunately, they cannot help us to get a stability result around solutions of (1.3) with a non trivial dynamics, since we shall prove that the only solutions (f, V) to (1.3) satisfying also (2.17) (along with some weak regularity assumptions) are the stationary equilibria μ for which $\bar{v}$ and φ are independent of t and x

$$\mu(v) = \varphi\left(-\frac{|v-\bar{v}|^2}{2}\right). \tag{2.18}$$

See Proposition 5.1 for the precise result. Loosely speaking, this means there is virtually no hope of relying on a modulated energy method to derive the quasineutral Vlasov equation (1.3), for non stationary data.

2.4 Construction of BGK waves for the quasineutral Vlasov equation.

In order to emphasize on the somewhat "degeneracy" of the limit system (1.3), we will also show that the construction of the equivalent of the so-called BGK waves in the Vlasov-Poisson case [11] is much more degenerate in the quasineutral case. Precisely, we will study the following boundary problem for the associated stationary kinetic equation

$$\begin{cases} v\,\partial_x f - \partial_x V \partial_v f = 0, \\ \rho = \int f(x,v)\,dv = 1, \end{cases} \tag{2.19}$$

on the space $\Omega = [0,1] \times \mathbb{R}$. The incoming boundary conditions are given by

$$\begin{cases} f(0,v) = f_0^+(v) & \text{if } v \geq 0, \\ f(1,v) = f_0^-(v) & \text{if } v \leq 0. \end{cases} \tag{2.20}$$

This model is the stationary equation associated to the quasineutral Vlasov equation (1.3), with the boundary conditions (2.20). We prove the following

Theorem 2.4. *Assume that $f_0^\pm : \mathbb{R}^\pm \to \mathbb{R}^+$ are two nonnegative and measurable functions such that $\int_0^\infty \big(f_0^+(v) + f_0^-(-v)\big)\,dv = 1$. Define the function f_T on $(0,+\infty)$ as below*

$$f_T(u) := \frac{1}{\pi}\int_0^\infty \big(f_0^+(v) + f_0^-(-v)\big)\frac{u\,v\,dv}{(u^2+v^2)^{\frac{3}{2}}}. \tag{2.21}$$

Then for any continuous potential $V : [0,1] \to \mathbb{R}^-$ satisfying $V(0) = V(1) = 0$ the function

$$(x,v) \mapsto f(x,v) = \begin{cases} f_0^+\big(\sqrt{v^2+2V(x)}\big) & \text{if} \quad v \geq \sqrt{-2V(x)}, \\ f_0^-\big(-\sqrt{v^2+2V(x)}\big) & \text{if} \quad v \leq -\sqrt{-2V(x)}, \\ f_T\big(\sqrt{-v^2-2V(x)}\big) & \text{if} \quad |v| < \sqrt{-2V(x)}, \end{cases} \tag{2.22}$$

together with V gives a solution of (2.19) in the sense of distributions. Moreover, any solution with V nonpositive and vanishing at the boundary is of the above form.

The striking points in Theorem 2.4 are that:

- there is a huge freedom in the choice of the potential V. In particular there is no a priori bound on its minimal value.
- the density of trapped function depends only on the boundary conditions $f_0^\pm$, and not on the potential V.

In some sense, this feature illustrates the degeneracy of the quasineutral Vlasov equation, compared to the classical Vlasov-Poisson. For instance, the fact that f_T is independent of V is due to the fact we pass from (2.19) to (2.23) by replacing a "local" equation by a Poisson equation. More precisely, the corresponding problem in the Vlasov-Poisson case is the construction of the so called BGK waves, which was performed in the pioneering work of Bernstein, Greene and Kruskal [11]. Consider the problem

$$\begin{cases} v\,\partial_x f_\varepsilon - \partial_x V_\varepsilon \partial_v f_\varepsilon = 0, \\ -\varepsilon^2 \partial_x^2 V_\varepsilon = \int f_\varepsilon \, dv - 1, \end{cases} \tag{2.23}$$

on the space $\Omega = [0,1] \times \mathbb{R}$, with the same boundary conditions (2.20). Building on [11], we can also construct numerous solutions $(f_\varepsilon, V_\varepsilon)$ of (2.23)- (2.20), but in this case, the density of trapped particles $f_{T,\varepsilon}$ always depends on the potential V_ε, and this one cannot be completely arbitrary: for instance it has to be bounded from below by some constant $C\varepsilon^{-2}$.

BGK waves play an important role in the large time dynamics of the Vlasov-Poisson equation: we refer on this topic to the works of Lin and Zheng [40, 41]. The abundance of BGK waves for the quasineutral Vlasov equation (1.3) shown in Theorem 2.4 suggests that the dynamics for (1.3) is very rich.

2.5 The case of the Vlasov-Poisson equation for ions.

The following Vlasov-Poisson equation

$$\left\{ \begin{array}{l} \partial_t f_\varepsilon + v\,\partial_x f_\varepsilon - \partial_x V_\varepsilon \, \partial_v f_\varepsilon = 0, \\ \alpha V_\varepsilon - \varepsilon^2 \partial_x^2 V_\varepsilon = \rho_\varepsilon - 1, \end{array} \right. \tag{2.24}$$

for some $\alpha > 0$, is also worth studying. Such a model, which we shall call *Vlasov-Poisson equation for ions* is often encountered in plasma physics: it aims at describing the dynamics of ions in a one dimensional plasma, when the electrons are assumed to be adiabatic (or massless), so that they have reached a thermodynamic equilibrium. When $\varepsilon \to 0$, one formally obtains the Vlasov-Dirac-Benney equation (1.8). This will also be studied later in this paper. We will explain in the last section how to adapt most of our results to that case.

As a final remark, let us mention that we expect that most of the results stated here have analogous statements in higher dimensions, up to some adaptations. For instance, for what concerns the unstable case, Sobolev spaces with weights in velocity should be useful. Such an adaptation could be the goal of some future works.

2.6 Organization of the following of the paper.

The following is dedicated to the proofs of our main results. In Section 3, we deal with the unstable case and provide a proof of Theorem 2.1. In Section 4 we study briefly the properties of the Casimir functional, we heuristically derive the equation for the correctors that are necessary in the ill-prepared case, and finally prove the stability Theorems 2.2 and 2.3. In Section 5, we prove that the only solutions to (1.3) satisfying locally everywhere the monotonicity and symmetry condition (2.17) are necessarily stationary. Then, in Section 6, we turn to the construction of BGK waves for (1.3) and prove Theorem 2.4. Finally, we explain how the results of this paper can be adapted to handle the Vlasov-Poisson equation for ions with adiabatic electrons in Section 7.

3 Unstable case: proof of Theorem 2.1

In this section we will first give some elements about the linearized Vlasov-Poisson equation (1.1) around homogeneous equilibria, and explain the relevancy of the Penrose criterion of Definition 1.1. Then, we shall provide a complete proof of Theorem 2.1.

That proof will rely on a instability theorem (Theorem 3.1) for the original Vlasov-Poisson equation (1.10) that is interesting by its own. It is proved in subsection 3.3. Later, we introduce the δ'-condition, and explain the necessary modifications to perform in the proof of Theorem 3.1.

We will end this section by proving Proposition 2.1.

3.1 Linearized Vlasov-Poisson equation and Penrose criterion

In this paragraph, we work on $\mathbb{T}_M \times \mathbb{R}$, where we recall that $\mathbb{T}_M := \mathbb{R}/(M\mathbb{Z})$ and $M > 0$. Given some smooth homogeneous equilibrium $\mu(v)$, we study the linearized Vlasov-Poison equation around μ:

$$\begin{cases} \partial_t f + Lf := \partial_t f + v\,\partial_x f - \partial_x V\,\mu' = 0, \\ -\partial_x^2 V = \displaystyle\int f dv, \end{cases} \quad \text{for } \ t \geq 0, x \in \mathbb{T}_M,\, v \in \mathbb{R}, \tag{3.1}$$

with $D(L) = \{f \in L^1_{x,v}(\mathbb{T}_M \times \mathbb{R}),\, \int f\, dvdx = 0,\, Lf \in L^1_{x,v}(\mathbb{T}_M \times \mathbb{R})\}$.

We shall rely on the description of the spectrum of the linearized Vlasov-Poisson equation, which was performed by Degond in [20]. We gather some useful information from [20, Theorem 1.1] in the following proposition.

Proposition 3.1. *Assume that $\mu \in W^{3,1}$. Consider the following dispersion relation, defined for $\lambda \in \mathbb{C} \setminus i\mathbb{R}$ and $n \in \mathbb{Z}^*$:*

$$D(n, \lambda) = 1 - \frac{M^2}{(2\pi n)^2} \int_{\mathbb{R}} \frac{\mu'(v)}{v - i\frac{M\lambda}{2\pi n}}\, dv. \tag{3.2}$$

i) The spectrum of L is given by

$$\sigma(L) = i\mathbb{R} \cup \{\lambda \in \mathbb{C},\, \exists n \in \mathbb{Z}^*, D(n, \lambda) = 0\}.$$

It is symmetric with respect to the real and imaginary axis. Moreover, λ is an eigenvalue of L if and only if $\lambda = 0$ or if there exists $n \in \mathbb{Z}^*$ such that $D(n, \lambda) = 0$.

Moreover, there exists $\omega_0 > 0$ such that $\sigma(L) \subset \{\lambda \in \mathbb{C}, |\Re\lambda| \leq \omega_0\}$.

ii) If $\Re\lambda > 0$, the set of solutions n of the equation $D(n, \lambda) = 0$ is finite and denoted by $\{n_1, \cdots, n_p\}$. In addition, a basis of the eigenspace associated to λ is given by

$$\left\{ e^{i\frac{2\pi n_j}{M}x} \frac{\mu'(v)}{v - i\frac{M\lambda}{2\pi n}},\, j = 1, \cdots, p \right\}. \tag{3.3}$$

In particular, these eigenfunctions associated to λ belong to $W^{s-1,1}$ if $\mu \in W^{s,1}$, and their associated spatial densities are equal to $\frac{(2\pi n)^2}{M^2} e^{i\frac{2\pi n_j}{M}x}$.

iii) Assume that μ is smooth. For all $\Gamma > \max\{\Re\lambda,\ \lambda \in \sigma(L)\}$ and all all $s \in \mathbb{N}$, there exists $C^s_\Gamma \geq 1$ such that the following holds, for all $h_0 \in W^{1,s}(\mathbb{T}_M \times \mathbb{R})$,

$$\forall t \geq 0, \quad \|e^{-tL} h_0\|_{W^{1,s}} \leq C^s_\Gamma\, e^{\Gamma t}\, \|h_0\|_{W^{1,s}}. \tag{3.4}$$

The proof of point $i)$ and $ii)$ in this Proposition is mostly done in [20, Theorem 1.1]. The remarks about the regularity and the spatial density in point $ii)$ are plain consequences of the particular form of the eigenfunctions, and of the dispersion relation (3.2). The estimate (3.4) is a consequence of the fact that L generates a strongly continuous semi-group on L^1 and of a standard bootstrap argument. The regularity assumptions on μ are not optimal, see [20] for details.

Remark that $D(n, \lambda) = 0$ may be rewritten as

$$G\Big(i\frac{M\lambda}{2\pi n}\Big) = \frac{(2\pi n)^2}{M^2}, \qquad \text{with} \quad G(\zeta) := \int_{\mathbb{R}} \frac{\mu'(v)}{v-\zeta}\, dv. \tag{3.5}$$

And in particular, there exists a eigenvalue λ with positive real part if and only if

$$\Big\{ \Big(\frac{2\pi n}{M}\Big)^2,\ n \in \mathbb{N} \Big\} \bigcap G(\Im^+) \neq \emptyset, \quad \text{where} \quad \Im^+ := \{z \in \mathbb{C},\ \text{s.t. } \Im z > 0\}. \tag{3.6}$$

As Penrose remarks in [49], the linear instability is therefore linked to the values taken by G on $\Im^+$, and since G is holomorphic, we can use some powerful theorems of complex analysis to simplify the condition (3.6). Before going on, we shall gather some properties of G in the following lemma.

Lemma 3.1. *Assume that $\mu \in W^{3,1}$. Then the function G is holomorphic on $\Im^+$. It can be extended to the real line by*

$$\forall \xi \in \mathbb{R}, \quad G(\xi) := P.V. \int_{\mathbb{R}} \frac{\mu'(v)}{v-\xi}\, dv + i\mu'(\xi), \tag{3.7}$$

where P.V. means that the integral has to be understood as a principal value. Moreover, the extended function G is uniformly continuous and $G(\xi)$ goes to zero when $|\xi|$ goes to infinity.

We shall not prove this lemma. We refer to [49] for some explanations. The uniform continuity is not stated in the later reference, but it is a consequence on the continuity of G on $\mathbb{R}$, which can be obtained after a careful estimate of the quantities involved.

In view of standard results of complex analysis, a complex z belongs to $G(\Im^+)$ if and only if it is encircled clockwise by the curve $G(\mathbb{R})$, covered form $-\infty$ to $+\infty$. In view of (3.6), there will be a eigenvalue with positive real part if and only if the curve $G(\mathbb{R})$ encircles clockwise a value $\big(\frac{2\pi n}{M}\big)^2$, for some $n \in \mathbb{N}$. But, as Penrose explains [49], it is possible only if the curve $G(\mathbb{R})$ crosses the half-line $\big(\big[\frac{2\pi n}{M}\big]^2, +\infty\big)$ from below at some point $G(\xi_0)$ for some $\xi_0 \in \mathbb{R}$. In view of the definition (3.7), ξ_0 has to be a strict and local minimum of μ, in the precise sense given in Definition 1.1. Moreover, the real part of $G(\xi_0)$ should be greater than $\big(\frac{2\pi n}{M}\big)^2$. But since $\mu'(\xi_0) = 0$, the principle value in (3.7) is a true integral, and it leads after an integration by parts to the necessary condition (1.12).

Nevertheless, this necessary condition is not sufficient: indeed, the curve $G(\mathbb{R})$ can cross by above the half-line of positive real numbers before surrounding one of the requested values. As mentioned in the introduction, it can be shown that the condition (1.12) is sufficient if the profile μ is symmetric with respect to the minimum (see [33]). But this symmetry condition is quite restrictive from the physical point of view. For instance, it is violated for what is usually called a “bump on tail” profile: a small bump added in the tail (for large velocities) of a given stable profile.

However, in the limit of large boxes (or equivalently in the limit of small Debye length), we will prove in the following proposition that the Penrose criterion of Definition 1.1 becomes a necessary and sufficient conditions for the existence of a eigenvalue with positive real part.

Proposition 3.2. *Assume that $\mu \in W^{3,1}$ satisfies the Penrose criterion of Definition 1.1. Then there exists a $\eta > 0$ such that if $\frac{1}{M} < \eta$, then the linearized operator L on $\mathbb{T}_M \times \mathbb{R}$ possesses an eigenvalue λ with $\Re\lambda > 0$. Moreover, for such an eigenvalue, there exists an associated eigenfunction h^λ of the form (3.3). In particular, it has some spatial inhomogeneity: precisely its associated density ρ^λ has a non zero real part. Finally, if $\mu \in W^{s,1}$, then $h^\lambda \in W^{s-1,1}$.*

Proof of Proposition 3.2. The proof relies only on elementary considerations. We shall treat only the case where the Penrose instability criterion is satisfied at a strict minimum. The case of a flat minimum can be handled in a similar fashion.

Choose a minimum point of μ, denoted by ξ_0, satisfying the Penrose criterion of Definition 1.1. As said before, it means that the curve $G(\mathbb{R})$ crosses at ξ_0 the half-line of positive real numbers, at the point $G(\xi_0)$. We set $\varepsilon := \frac{1}{2}G(\xi_0)$. Since the extended function G is uniformly continuous by Lemma 3.1, we can choose some $\eta > 0$ such that $|G(\xi) - G(\xi')| \leq \varepsilon$ when $|\xi - \xi'| \leq \eta$. We choose also two real numbers ξ_- and ξ_+ satisfying

$$\xi_0 - \eta \leq \xi_- < \xi_0 < \xi_+ \leq \xi_0 + \eta, \quad \text{and} \quad \Im G(\xi_-) < 0, \qquad \Im G(\xi_+) > 0.$$

It is possible since ξ_0 is a strict local minimum of μ. Remark that thanks to the definition of η, they also satisfy $\Re G(\xi_\pm) \geq \frac{1}{2}G(\xi_0)$.

Next, we define $\varepsilon' = \min\big(\frac{1}{2}G(\xi_0), |\Im G(\xi_-)|, |\Im G(\xi_+)|\big)$, and associate to it some $\eta' > 0$ by uniform continuity of G. Then, we have $\Re\, G(\xi_- + i\eta) > 0$ and $\Im\, G(\xi_- + i\eta) < 0$ and similarly $\Re\, G(\xi_+ + i\eta) > 0$ and $\Im\, G(\xi_+ + i\eta) > 0$. Moreover we have $G([\xi_-, \xi_+] + i\eta) \subset \{\Re z > 0\}$. By the intermediate value theorem, it means that there exists some $\xi_1 \in [\xi_-, \xi_+]$ such that $G(\xi_1 + i\eta) \in (0, +\infty)$.

But, since G is holomorphic on $\Im^+$, its image $G(\Im^+)$ is open, and we can therefore conclude that $G(\Im^+)$ contains some interval (a, b) with $0 < a < b$. Then, one can readily see that for M large enough

$$\Big\{ \Big(\frac{2\pi n}{M}\Big)^2,\ n \in \mathbb{N}\Big\} \bigcap (a, b) \neq \emptyset,$$

so that condition (3.6) is satisfied. The existence of a eigenvalue with positive part follows, and the claimed properties of the associated eigenfunction are a direct consequence of Proposition 3.1. $\square$

3.2 Proof of Theorem 2.1

Let $\mu(v)$ be a smooth profile satisfying the instability criterion of Definition 1.1. Using Proposition 3.2, we fix $M > 0$ large enough such that the linearized operator L around μ on $\mathbb{T}_M \times \mathbb{R}$ possesses a eigenvalue λ with positive real part.

From now on, we consider only the sequence $\varepsilon_k = \frac{1}{kM}$, for $k \in \mathbb{N}^*$, but we shall forget the k subscript for readability.

Step 1. Highly oscillating data.

We consider εM-periodic (in x) solutions to (1.1). Precisely, we look at solutions to the system

$$\begin{cases} \partial_t \tilde{f}_\varepsilon + v\, \partial_x \tilde{f}_\varepsilon - \partial_x \tilde{V}_\varepsilon\, \partial_v \tilde{f}_\varepsilon = 0, \\ -\varepsilon^2 \partial_x^2 \tilde{V}_\varepsilon = \displaystyle\int \tilde{f}_\varepsilon dv - 1, \end{cases} \quad \text{for } t \geq 0,\ x \in \mathbb{T}_{\varepsilon M} := \mathbb{R}/(\varepsilon M \mathbb{Z}),\ v \in \mathbb{R}. \tag{3.8}$$

We can canonically obtain from $\tilde{f}_\varepsilon$ a solution f_ε to (1.1) by "gluing" together $(\varepsilon M)^{-1}$ copies of $\tilde{f}_\varepsilon$.

Step 2. Rescaling.

We perform the change of variables $(t, x, v) \to \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v\right)$. In other words, we consider $(g_\varepsilon, \varphi_\varepsilon)$ such that:

$$\tilde{f}_\varepsilon(t, x, v) = g_\varepsilon\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v\right), \quad \tilde{V}_\varepsilon(t, x) = \varphi_\varepsilon\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right). \tag{3.9}$$

This leads to the study of the following system, which is now *independent of* ε, and posed for $t \geq 0$, $x \in \mathbb{T}_M$, $v \in \mathbb{R}$

$$\begin{cases} \partial_t g + v\,\partial_x g - \partial_x \varphi\,\partial_v g = 0, \\ -\partial_x^2 \varphi = \displaystyle\int g dv - 1, \end{cases} \quad \text{for } \ t \geq 0, x \in \mathbb{T}_M,\, v \in \mathbb{R}. \tag{3.10}$$

Remark that the standard Sobolev embedding on $\mathbb{T}_M$ implies a good control on the electric field. Precisely, for all $s \in \mathbb{N}$, if h has zero mean on $\mathbb{T}_M \times \mathbb{R}$, and φ satisfies $-\partial_x^2 \varphi = \int h dv$, then

$$\|\partial_x^{s+1} \varphi\|_\infty \leq \|\partial_x^s h\|_1. \tag{3.11}$$

We shall use that estimate at multiple times in what follows.

Linearizing (3.10) around μ, we obtain

$$\begin{cases} \partial_t h + Lh = \partial_t h + v\,\partial_x h - \partial_x \Psi\,\partial_v \mu = 0, \\ -\partial_x^2 \Psi = \displaystyle\int h dv, \end{cases} \quad \text{for } \ t \geq 0, x \in \mathbb{T}_M,\, v \in \mathbb{R}, \tag{3.12}$$

which is exactly the linearized system studied in the section 3.1.

Step 3. A nonlinear instability result.

The description of the spectrum obtained in the section 3.1 allows to deduce the following non-linear instability theorem on the rescaled system.

Theorem 3.1. *Assume that the profile μ satisfies the Penrose instability criterion* (1.11)*, the technical condition* (2.2) *(or the condition* (3.31)*) and belongs to all the $W^{s,1}$ for $s \in \mathbb{N}$. Then, there exists a sequence $(\theta_r)_{r\in\mathbb{N}}$ of positive real numbers such that for any $S \in \mathbb{N}$, and any $\delta > 0$, there exists a solution (g, φ) to* (3.10) *with positive g satisfying $\|g(0) - \mu\|_{W^{S,1}(\mathbb{T}_M\times\mathbb{R})} \leq \delta$ but such that*

$$\begin{aligned} &\theta_0 \leq \sup_{t\in[0,t_\delta]} \left\| \int_{\mathbb{R}} g(t, x, v)\, dv - 1 \right\|_{W_x^{-1,1}(\mathbb{T}_M)} \leq \sup_{t\in[0,t_\delta]} \left\| \int_{\mathbb{R}} g(t, x, v)\, dv - 1 \right\|_{L_x^1(\mathbb{T}_M)} \\ &\forall r \in \mathbb{N}^*, \quad \theta_r \leq \sup_{t\in[0,t_\delta]} \left\| \int_{\mathbb{T}_M} \Big(g(t, x, v) - \mu(v)\Big)\, dx \right\|_{W_v^{-r,1}}, \end{aligned} \tag{3.13}$$

with, for a fixed S, $t_\delta = O(|\log \delta|)$ as $\delta \to 0$.

Remark 3.1. *Assume that μ belongs only to the Sobolev space $W^{r,1}$, for some $r \in \mathbb{N}$. Then, the previous result is valid for $s \leq r - 1$ if the regularity index r satisfies the condition*

$$r > 1 + \frac{\|\partial_v \mu\|_1}{\max\{\Re\lambda, \lambda \in \sigma(L)\}},$$

where L is the linearized operator defined in (3.12).

The proof of Theorem 3.1 follows a method introduced by Grenier in [31] which is by now standard in instability theory for hydrodynamic equations. We postpone it to the next subsection, after the conclusion of the proof of Theorem 2.1.

Step 4. Back to the original variables.

Let $s, N \in \mathbb{N}^*$. Take any $P \in \mathbb{N}$, such that $P > s+N$. By Theorem 3.1, we find for all ε small enough a solution $(g_\varepsilon, \varphi_\varepsilon)$ to (3.10) satisfying $\|g_\varepsilon(0) - \mu\|_{W^{s,1}} \leq \varepsilon^P$ and (3.13). The associated instability time will be denoted $t_\varepsilon = O(|\log \varepsilon|)$, and the density by $\Lambda_\varepsilon(t) = \int_{\mathbb{R}} g_\varepsilon(t,x,v)\, dv$.

Next, a consequence of the εM-periodicity of f_ε and of the change of variable (3.9) is that (at any time t):

$$\begin{aligned} &\|\rho_\varepsilon - 1\|_{L^1} = \frac{1}{M}\|\Lambda_\varepsilon - 1\|_{L^1(\mathbb{T}_M)}, \qquad \|\rho_\varepsilon - 1\|_{W^{-1,1}} \geq \frac{\varepsilon}{C\,M}\|\Lambda_\varepsilon - 1\|_{W^{-1,1}(\mathbb{T}_M)}, \\ &\|f_\varepsilon - \mu\|_{W^{s,1}} \leq \frac{\varepsilon^{-s}}{M}\|g_\varepsilon - \mu\|_{W^{s,1}(\mathbb{T}_M\times\mathbb{R})} \quad \text{for } s \in \mathbb{N}. \end{aligned} \tag{3.14}$$

A constant $C > 0$ appears in the second inequality of the first line because we are using non-homogeneous Sobolev norms.

Since the velocity variable is not affected by the scaling, we also have

$$\|f_\varepsilon - \mu\|_{W^{-r,1}_{x,v}} \geq \left\| \int_{\mathbb{T}} (f_\varepsilon(t,x,v) - \mu(v))\, dx \right\|_{W_v^{-r,1}} = \frac{1}{M}\left\| \int_{\mathbb{T}_M} (g_\varepsilon(t,x,v) - \mu(v))\, dx \right\|_{W_v^{-r,1}}. \tag{3.15}$$

From this, we deduce that (for ε small enough)

$$\begin{aligned} &\|f_\varepsilon(0) - \mu\|_{W^{s,1}} \leq \frac{1}{M}\,\varepsilon^{P-s} \leq \varepsilon^N, \\ &\frac{\theta_0}{C\,M} \leq \sup_{t\in[0,\varepsilon t_\varepsilon]} \frac{1}{\varepsilon}\|\rho_\varepsilon - 1\|_{W^{-1,1}} \leq \sup_{t\in[0,\varepsilon t_\varepsilon]} \|\rho_\varepsilon(t) - 1\|_{L^1}, \\ &\theta_r \leq \sup_{t\in[0,\varepsilon t_\varepsilon]} \|f_\varepsilon(t) - \mu\|_{W^{-r,1}}. \end{aligned}$$

We therefore deduce the first point in $i)$ and $ii)$. From the bound from below on the $W^{-1,1}$ norm of $\rho_\varepsilon - 1$, using the Poisson equation in (1.1), we deduce the second point of $i)$. Finally note that $\varepsilon\, t_\varepsilon = O(\varepsilon\,|\ln \varepsilon|)$. We have thus completed the proof of Theorem 2.1.

Remark 3.2. *Remark that if μ is analytic, then the initial data $f_\varepsilon(0)$ we build are also analytic, because of $ii)$ in Proposition 3.1. Nevertheless, it is not possible to get a similar theorem with the $W^{1,s}$-norm replaced by some analytic norm. Indeed, in this case, we rather expect stability, at least for short times: see for instance [28, Theorem 1.1.2] for a stability result with the "superposition of fluids" point of view, and also the work [13] about the well-posedness of the quasineutral equation.*

We found it quite interesting to understand what in our proof prevents us from keeping analytic norms from start to finish. This is precisely due to Step 4., where the rescaling in space (and especially the factor ε^{-s} in the above bounds) prevents from getting "uniform" in ε analytic bounds.

3.3 Proof of Theorem 3.1

We use Proposition 3.2 and denote by h_1 an eigenfunction associated to an eigenvalue with maximal real part of L, denoted by λ_1 (with $\Re\lambda_1 > 0$), and such that $\rho_1 := \int h_1\, dv \neq 0$. Up to a multiplication by a constant, we may assume that $\|h_1\|_{L^1}$=1. Note that this implies that

$0 < \|\rho_1\|_{W^{-1,1}} \le \|\rho_1\|_1 \le 1$. Then, a good candidate for g is the solution of the Vlasov-Poisson equation (3.10) with initial condition

$$g(0) = \mu + \delta h_1,$$

since according to the study of the linearized operator, loosely speaking, this solution will remain close (for small time) to

$$g^1_{app}(t) := \mu + \delta e^{\Re \lambda_1 t} h_1,$$

and thus "escape" from any small neighborhood of μ. But the control of the errror between g and its linear approximation g^1_{app} on a sufficiently large time interval is not so straightforward: Grenier's method for overcoming this difficulty involves constructing a convenient high-order approximation of g.

Note that the initial datum $g(0)$ defined above is a priori complex valued, since h_1 is. But we will show first that this $g(0)$ satisfies all the requested properties except it is not real, and explain in the last step of the proof, how to obtain from this $g(0)$ a non-negative initial condition with the requested properties.

Step 1. A formal high order approximation. Precisely, we look for a series of functions g^N_{app} satisfying (3.10) up to a small remainder R^N_{app}

$$\partial_t g^N_{app} + v\,\partial_x g^N_{app} + \partial_x V^N_{app}\,\partial_v g^N_{app} = R^N_{app},$$

where as usual $V^N_{app} := \partial^{-1}_{xx}(\int g^N_{app}\,dv - 1)$. The initial condition is the same as g : $g^N_{app}(0) = g(0) = \mu + \delta h_1$.

The functions g^N_{app} will be constructed as the partial sum of a series, whose terms will be defined by induction in order to decrease the order of the remainder R^N_{app} at each step:

$$g^N_{app}(t,x,v) = \mu(v) + \delta h_1(x,v) e^{\lambda_1 t} + \sum_{i=2}^N \delta^i h_i(t,x,v).$$

We will also use the notation $h_1(t,x,v) = h_1(x,v)e^{\lambda_1 t}$. Starting with $N = 1$, we can see that $g^1_{app} = \mu + \delta h_1(x,v)e^{\lambda_1 t}$ is the solution to

$$\partial_t g^1_{app} + v\,\partial_x g^1_{app} + \partial_x V^1_{app}\,\partial_v g^1_{app} = \delta^2 E_1 \partial_v h_1 = R^1_{app},$$

with the notation $E_k := \partial_x \partial^{-1}_{xx}\big(\int h_k\,dv\big)$. In order to find the appropriate value of h_2, we can plug g_2 in the rescaled Vlasov-Poisson equation (3.10) and get

$$\begin{aligned}\partial_t g^2_{app} + v\,\partial_x g^2_{app} + \partial_x V^2_{app}\,\partial_v g^2_{app} &= \delta^2(\partial_t h_2 + L h_2 + E_1 \partial_v h_1)\\ &\quad + \delta^3(E_1\partial_v h_2 + E_2 \partial_v h_1) + \delta^4(E_2 \partial_v h_2).\end{aligned}$$

We see that the best choice for h_2 and more generally for h_k for $k \ge 2$ is to take it as the solution of

$$\partial_t h_k + L h_k + \sum_{j=1}^{k-1} E_j \partial_v h_{k-j} = 0, \tag{3.16}$$

with $h_k(0) = 0$ as initial datum. Of course, this can be done only if the h_j are regular enough, but we will check this fact later. Then, the associated remainder term R^N_{app} is given by

$$R^N_{app} = \sum_{N+1 \le j+j' \le 2N} \delta^{j+j'} E_j \partial_v h_{j'}. \tag{3.17}$$

Remark also that in view of (3.4) and the form of the source term in (3.16) (and up to some regularity issue), we allow $\|h_k\|_1$ to grow at most like $e^{k\Re\lambda_1 t}$. This implies that the remainder $\|R^N_{app}\|_1$ will grow at most like $\delta^{N+1}e^{(N+1)\Re\lambda_1 t}$, for not too large times.

Step 2. A heuristic error estimate. Our first goal is to obtain good estimates on $\|g - g^N_{app}\|_1$. To this end, remark that $(g - g^N_{app})$ is solution to

$$\partial_t(g - g^N_{app}) + v\partial_x(g - g^N_{app}) + \partial_x V\partial_v(g - g^N_{app}) = (\partial_x V^N_{app} - \partial_x V)\partial_v g^N_{app} - R^N_{app}.$$

If we multiply this equation by $\mathrm{sign}(g - g^N_{app})$ and integrate with respect to x and v, we get

$$\begin{aligned}\frac{d}{dt}\|g - g^N_{app}\|_1 &\le \|\partial_x V^N_{app} - \partial_x V\|_\infty \|\partial_v g^N_{app}\|_1 + \|R^N_{app}\|_1,\\ &\le \|\partial_v g^N_{app}\|_1 \|g - g^N_{app}\|_1 + \|R^N_{app}\|_1.\end{aligned} \tag{3.18}$$

Assume that we are able to control $\|\partial_v(g^N_{app} - \mu)(t)\|_1 \le 1$ on a time interval $[0,T]$. It is reasonable to expect such a control since δ is small and we will show that the h_k are smooth enough. Then, for $t \in [0,T]$, we obtain a bound

$$\|g(t) - g^N_{app}(t)\|_1 \le \int_0^t e^{(t-s)(\|\partial_v\mu\|_1+1)}\|R^N_{app}(s)\|_1\, ds. \tag{3.19}$$

If we take for granted the expected growth of the remainder $\|R^N_{app}\|_1 \lesssim \delta^{N+1}e^{(N+1)\Re\lambda_1 t}$, we get an estimate

$$\begin{aligned}\|g(t) - g^N_{app}(t)\|_1 &\lesssim \delta^{N+1}e^{\max((N+1)\Re\lambda_1, \|\partial_v\mu\|_1+1))t},\\ &\lesssim \left[\delta e^{\Re\lambda_1 t}\right]^{N+1} \qquad \text{if } (N+1)\Re\lambda_1 > \|\partial_v\mu\|_1 + 1.\end{aligned}$$

The last bound will be very important for the following argument, since it allows to compare $\|g(t) - g^N_{app}(t)\|_1$ to $\|g^N_{app}(t) - \mu(t)\|_1 \approx \delta e^{\Re\lambda_1 t}$, almost independently of the time. So from now on, we fix (with the notation $\lfloor\cdot\rfloor$ for the integer part)

$$N := \left\lfloor \frac{\|\partial_v\mu\|_1 + 1}{\Re\lambda_1} \right\rfloor. \tag{3.20}$$

In fact, the factor 1 added to the norm of $\partial_v\mu$ may be replaced by any positive real number, and that leads to the condition given in Remark 3.1.

Step 3. A rigorous error estimate. By assumption, we know that $\mu \in W^{N+1,1}$. By Proposition 3.2, $h_1 \in W^{N,1}$, and thanks to its definition, $\|h_1(t)\|_{W^{N,1}} = \|h_1\|_{W^{N,1}}e^{\Re\lambda_1 t}$. We now show by recursion that for all $k \le N$ there exists a constant $C_k > 0$ such that for any time $t \ge 0$,

$$\|h_k(t)\|_{W^{N-k+1,1}} \le C_k e^{k\Re\lambda_1 t}.$$

By construction, this is true for $k = 1$. We choose a $\Gamma \in (\Re\lambda_1, 2\Re\lambda_1]$, and we assume the bound holds until rank k. Then for the rank $k+1$, by the definition (3.16), the estimate (3.4) on the semi-group generated by L, and with the help of Duhamel's formula, we get

$$\begin{aligned}
\|h_{k+1}\|_{W^{N-k,1}} &\leq \sum_{j=1}^{k} \int_0^t \|e^{(t-s)L}\big(E_j \partial_v h_{k+1-j}\big)\|_{W^{N-k,1}} ds \\
&\leq C_\Gamma^{N-k} \sum_{j=1}^{k} \int_0^t e^{\Gamma(t-s)} \|E_j\|_{W^{N-k,\infty}} \|\partial_v h_{k+1-j}\|_{W^{N-k,1}}\, ds \\
&\leq C_\Gamma^{N-k} \sum_{j=1}^{k} \int_0^t e^{\Gamma(t-s)} \|h_j\|_{W^{N-k,1}} \|h_{k+1-j}\|_{W^{N-k+1,1}}\, ds \\
&\leq C_\Gamma^{N-k} \Big(\sum_{j=1}^{k} C_j C_{k+1-j}\Big) \int_0^t e^{\Gamma(t-s)} e^{(k+1)\Re\lambda_1 s} ds \\
&\leq C_{k+1}\, e^{(k+1)\Re\lambda_1 t}.
\end{aligned}$$

Remark that we have used the Sobolev embedding (3.11). By formula (3.17), we obtain also, for the remainder, the bound

$$\|R^N_{app}(t)\|_{L^1} \leq C'_N\, \delta^{N+1} e^{(N+1)\Re\lambda_1 t}, \qquad \text{as long as } \ \delta e^{\Re\lambda_1 t} < 1. \tag{3.21}$$

Step 4. Instability with complex valued approximation. We can now estimate $\|g^N_{app} - g^1_{app}\|_1$ and the term $\|\partial_v g^N_{app}\|_1$ that appears in (3.18). We introduce with $C''_N = \max_{k\leq N} C_k$

$$\theta_{max} := \frac{\|\rho_1\|_{W^{-1,1}}}{3\max(1, C'_N, C''_N)}, \qquad t_{max} := \frac{1}{\Re\lambda_1} \ln\Big(\frac{\theta_{max}}{\delta}\Big),$$

so that $\delta e^{\Re\lambda_1 t} \leq \theta_{max}$ if and only if $t \leq t_{max}$. Then,

$$\begin{aligned}
\|g^1_{app}(t) - g^N_{app}(t)\|_1 &\leq \sum_{k=2}^{N} \delta^k \|h_k(t)\|_1 \leq \sum_{k=2}^{N} C_k \big[\delta e^{\Re\lambda_1 t}\big]^k \\
&\leq C''_N \frac{\big[\delta e^{\Re\lambda_1 t}\big]^2}{1-\delta e^{\Re\lambda_1 t}} \leq \frac{\delta}{2}\, \|\rho_1\|_{W^{-1,1}}\, e^{\Re\lambda_1 t}, \qquad \text{for } t \leq t_{max}.
\end{aligned} \tag{3.22}$$

Similarly,

$$\begin{aligned}
\|\partial_v(g^N_{app}(t) - \mu)\|_1 &\leq \sum_{k=1}^{N} \delta^k \|\partial_v h_k\|_1 \leq \sum_{k=1}^{N} C_k \delta^k e^{k\,\Re\lambda_1 t} \\
&\leq C''_N \frac{\delta e^{\Re\lambda_1 t}}{1-\delta e^{\Re\lambda_1 t}} \leq \frac{1}{2} \qquad \text{as long as } t \leq t_{max}.
\end{aligned}$$

As a consequence, for $t \leq t_{max}$, we can apply (3.19) and using estimate (3.21) and definition (3.20), it comes

$$\begin{aligned}
\|g(t) - g^N_{app}(t)\|_1 &\leq C'_N \int_0^t e^{(t-s)(\|\partial_v\mu\|_1+1)}\, \delta^{N+1} e^{(N+1)\Re\lambda_1 s}\, ds, \\
&\leq C'_N \big[\delta e^{\Re\lambda_1 t}\big]^{N+1} = C'_N \big[\theta_{max}\big]^N \delta\, e^{\Re\lambda_1 t} \leq \frac{\delta}{3}\, \|\rho_1\|_{W^{-1,1}}\, e^{\Re\lambda_1 t}.
\end{aligned} \tag{3.23}$$

Gathering (3.22) and (3.23) we obtain a good control of the error in the approximation of g by g^1_{app}:

$$\|g(t) - g^1_{app}(t)\|_1 \le \frac{5\delta}{6}\, \|\rho_1\|_{W^{-1,1}}\, e^{\Re\lambda_1 t}, \qquad \text{for } t \le t_{max}. \tag{3.24}$$

This implies that for $t \le t_{max}$

$$\begin{aligned} \left\| \int g(t)\, dv - 1 \right\|_{W^{-1,1}} &\ge \left\| \int g^1_{app}(t)\, dv - 1 \right\|_{W^{-1,1}} - \|g(t) - g^1_{app}(t)\|_1 \\ &\ge \delta\, \|\rho_1\|_{W^{-1,1}}\, e^{\Re\lambda_1 t} - \frac{5\delta}{6} \|\rho_1\|_{W^{-1,1}}\, e^{\Re\lambda_1 t} = \frac{\delta}{6} \|\rho_1\|_{W^{-1,1}}\, e^{\Re\lambda_1 t}. \end{aligned} \tag{3.25}$$

So in particular, at $t = t_{max}$, we have $\left\| \int g(t)\, dv - \mu \right\|_1 \ge \frac{1}{6}\theta_{max} \|\rho_1\|_{W^{-1,1}}$. Thus, with the notation of Theorem 3.1, we can choose $\theta_0 = \frac{1}{6}\theta_{max}\|\rho_1\|_{W^{-1,1}}$.

For what concerns the initial condition, for a fixed s, we have

$$\|g(0) - \mu\|_{W^{s,1}} = \delta\, \|h_1\|_{W^{1,s}} = C_s\, \delta,$$

since μ belongs to all the $W^{s,1}$ spaces by assumption and so does h_1 by Proposition 3.2. From this, we see that $\|g(0) - \mu\|_{W^{s,1}}$ can be made as small as we want, and that in any case the density $\int g(t)\, dv$ will move away from 1 by a distance of at least θ_0 in $W^{-1,1}$-norm, and this before a time of order $\ln\big(\|g(0) - \mu\|^{-1}_{W^{s,1}}\big)$.

Step 5. The construction of a real initial condition. In order to construct a *real* initial condition with the requested properties, we remark that $\bar{h}_1$ is also an eigenfunction of L, associated to the eigenvalue $\bar{\lambda}_1$, and thus we choose as new initial condition

$$\tilde{g}(0) := \mu + \frac{\delta}{2}\, [h_1 + \overline{h_1}]. \tag{3.26}$$

Remark that $\Re h_1 \neq 0$ (recall Proposition 3.2). Then, using Grenier's method with this starting point, it can be shown that the solution $\tilde{g}$ to (1.1) starting form $\tilde{g}(0)$ has the same properties as those of g. The same proof can be performed again (we shall now write it again for the sake of conciseness), by considering similar functions (we will systematically add a tilde when we will refer to them in the following). We just remark that we should replace $\|h_1\|_1$ by $\|\Re h_1\|_1$, $\|\rho_1\|_{W^{-1,1}}$ by $\|\Re\rho_1\|_{W^{-1,1}}$, and that in (3.25), the term $\int \tilde{g}^1_{app}(t)\, dv$ will now oscillate; but the bound by above is still true if t is a multiple of 2π, and this is sufficient to conclude.

In the case were μ satisfies the δ-condition of Definition 2.1, then the particular form of h_1 given in (3.3) implies that $\delta|h_1| \le \mu$ for δ small enough, and then that the $\tilde{g}(0)$ defined in (3.26) is non-negative.

But nothing ensures that the $\tilde{g}$ constructed above is nonnegative when μ satisfies only the δ'-condition of Definition 3.1. This will require some truncation argument, which will be performed in the next subsection, after the precise statement of the δ'-condition.

Before, let us now study the instability in $W^{-r,1}_v$ for the averages in x.

Step 6. The instability in $W^{-r,1}_v$ on the average in position. By Proposition 3.1, $\tilde{h}_1(t,x,v)$ and its associated density $\tilde{\rho}_1$ are of the form below for some $n \in \mathbb{Z}^*$, $\xi \in \mathbb{C}\backslash\mathbb{R}$ and $\kappa > 0$

$$\tilde{h}_1(t,x,v) = \frac{1}{2}\Big(e^{\lambda_1 t + i\frac{2\pi n}{M}x}\, \frac{\mu'(v)}{v+\xi} + e^{\overline{\lambda_1} t - i\frac{2\pi n}{M}x}\, \frac{\mu'(v)}{v+\bar{\xi}} \Big), \qquad \tilde{\rho}_1(t,x) = \kappa\, e^{\Re\lambda_1 t} \cos\left(\Im\lambda_1 t + \frac{2\pi n}{M} x \right). \tag{3.27}$$

The fact that κ is a positive real number is a consequence of the dispersion relation (3.2). Remark that x-average of $\tilde{h}_1$ vanishes. Thus, in order to see an instability on the x-average we should also study $\tilde{h}_2$. But $\tilde{h}_2$ is solution to

$$\partial_t \tilde{h}_2 + L\tilde{h}_2 + \tilde{E}_1 \, \partial_v \tilde{h}_1 = 0.$$

Integrating with respect to x, and using that $\int L\tilde{h}_2 \, dx = 0$, we get

$$\partial_t \Big(\int \tilde{h}_2(t,x,v) \, dx \Big) = - \int \tilde{E}_1(t,x) \, \partial_v \tilde{h}_1(t,x,v) \, dx.$$

Using (3.27) and $\partial_x \tilde{E}_1 = \tilde{\rho}_1$, we get $\tilde{E}_1 = \frac{\kappa M}{2\pi n} e^{\Re \lambda_1 t} \sin\big(\Im \lambda_1 t + \frac{2\pi n}{M} x\big)$, and this allows to calculate the r.h.s. in the last equation. After a short calculation and a integration in time, we get

$$\int_{\mathbb{T}_M} \tilde{h}_2(t,x,v) \, dx = - \frac{\kappa M}{4\pi n \, \Re \lambda_1} \big[e^{2\Re \lambda_1 t} - 1 \big] \ell'(v), \tag{3.28}$$

where ℓ is a smooth function defined by

$$\ell(v) := \Im \Big[\frac{\mu'(v)}{v + \xi} \Big] = \frac{\Im \xi \, \mu'(v)}{(v + \Re \xi)^2 + (\Im \xi)^2}.$$

In particular, we have for any $r \in \mathbb{N}$

$$\begin{aligned} \Big\| \int \tilde{h}_2(t,x,v) \, dx \Big\|_{W_v^{-r,1}} &= \sup_{\|\varphi\|_{W^{r,\infty}} \leq 1} \int \tilde{h}_2(t,x,v) \varphi(v) \, dx \, dv \\ &\geq \frac{1}{\|\ell\|_{W^{r+1,\infty}}} \int \tilde{h}_2(t,x,v) \ell'(v) \, dx \, dv \\ &= \frac{\kappa M \|\ell'\|_2^2}{4\pi n \, \Re \lambda_1 \, \|\ell\|_{W^{r+1,\infty}}} \big[e^{2\Re \lambda_1 t} - 1 \big] =: c_r' \big[e^{2\Re \lambda_1 t} - 1 \big]. \end{aligned}$$

In particular, remark that since c_r' is a well-defined constant since ℓ is as smooth as μ' and also $\|\ell'\|_2$ is finite since $\mu'' \in L^1 \cap L^\infty$. Therefore, the previous bound by below leads to

$$\Big\| \int \Big(\tilde{g}^2_{app}(t,x,v) - \mu(v) \Big) \, dx \Big\|_{W_v^{-r,1}} \geq c_r' \delta^2 \, \big[e^{2\Re \lambda_1 t} - 1 \big].$$

Starting from this inequality, the strategy of the Step 4 can be performed again, and we can obtain (up to some redefinition of θ_{max} and t_{max}) the conclusion claimed in Theorem 3.1. In fact all the remainder terms $\tilde{g} - \tilde{g}^N_{app}$ and $\tilde{g}^N_{app} - \tilde{g}^2_{app}$ are controlled without integration in x, in a stronger topology (namely L^1) and at a smaller order (at most $[\delta e^{\Re \lambda_1 t}]^3$).

This conclude the proof in the case where μ satisfies the δ-condition of Definition 2.1.

3.4 The alternative δ'-condition.

Definition 3.1. *For any non-negative C^1 profile $\mu(v)$, and any $\delta > 0$, we define*

$$V_\delta := \Big\{ v \in \mathbb{R}, \ \ s.t. \ \frac{|\mu'(v)|}{1 + |v|} > \frac{1}{\delta} \mu(v) \Big\} \quad \subset \mathbb{R}, \tag{3.29}$$

$$W_\delta := \{ w \in \mathbb{R}, \ \ s.t. \ d(w, V_\delta) \leq \sqrt{\delta} \}, \tag{3.30}$$

where d stand for the usual distance from a point to a set. We say that μ satisfies the δ'-condition if for any $n \in \mathbb{N}$,

$$\liminf_{\delta \to 0} \frac{1}{\delta^n} \int_{W_\delta} |\mu'(v)| \, dv = 0. \tag{3.31}$$

With that new condition, the conclusion of Theorem 2.1 still holds. We provide in the step 7 belows the truncation argument (see also [34] for a similar construction).

Step 7 of the proof of Theorem 2.1. The construction of a non-negative initial condition.

Our goal is now to show how to construct a relevant non-negative initial condition. Recall that from Proposition 3.1, the eigenfunction satisfies for some $m \in \mathbb{N}$, $\xi \in \mathbb{C}\backslash\mathbb{R}$ and thus for some $C_1 \geq 1$

$$|h_1(x,v)| = \left| e^{imx} \frac{\mu'(v)}{v+\xi} \right| \leq C_1 \frac{|\mu'(v)|}{1+|v|}.$$

Remark that the real initial condition $\tilde{g}(0)$ defined in a previous step may take negative value at any point v where

$$\delta |h_1|(v) > \mu(v).$$

In order to "remove" such problematic points, we introduce a smooth cut-off function $k : \mathbb{R} \to [0,1]$ such that $k(x) = 0$ for $x \leq 0$, $k(x) = 1$ when $x \geq 1$, and define

$$V'_\delta := \left\{ v \in \mathbb{R}, \ \text{ s.t. } |h_1|(v) > \frac{1}{\delta}\mu(v) \right\},$$

and its $\sqrt{\delta}$-neighborhood W'_δ. Remark that V'_δ is related to V_δ defined in 3.29: precisely we have $V_\delta \subset V_{C_1\delta}$ and $W_\delta \subset W_{C_1\delta}$ since $C_1 > 1$, so that the property (3.31) is still true with W_δ replaced by W'_δ.

We also define $G'_\delta := \{w, \ \text{ s.t. } d(w, V'_\delta) \geq \frac{1}{2}\sqrt{\delta}\}$, and denote by χ_δ its characteristic function. Then we choose η a smooth function with total mass one and support in $[-1,1]$, and define for any $\delta > 0$, $\eta_\delta := 2\delta^{-\frac{1}{2}}\eta\big(\frac{1}{2}\sqrt{\delta}\,\cdot\big)$, which has still total mass one and a support in $\big[-\frac{1}{2}\sqrt{\delta}, \frac{1}{2}\sqrt{\delta}\big]$. Then we define

$$h_1^\delta(x,v) := h_1(x,v)\, [\chi_\delta * \eta_\delta](v).$$

Then h_1^δ satisfies the following properties:

a) $h_1^\delta = h_1$ on $\mathbb{R}\backslash W'_\delta$;

b) $h_1^\delta = 0$ on V'_δ, so that $|h_1^\delta| \leq \delta\mu$;

c) for any $s \in \mathbb{N}$,

$$\|h_1^\delta\|_{W^{s,1}_{x,v}} \leq C\delta^{-\frac{s}{2}} \|h_1\|_{W^{s,1}_{x,v}}. \tag{3.32}$$

From now on, we fix $n = \max(N+1, S)$. In view of (3.31) satisfied by μ, there exists a sequence of positive number $(\delta_k)_{k\in\mathbb{N}}$ converging to 0 such that

$$\lim_{k\to+\infty} \frac{1}{\delta_k^n} \int_{v\in W'_{\delta_k}} |\mu'(v)|\, dv = 0.$$

From now on, we assume that δ take only the values δ_k of that sequence, but do not write the index k for readability. We have using point $a)$ above and (3.31)

$$\|h_1 - h_1^\delta\|_{L^1} \leq C_1 \int_{v\in W'_\delta} |\mu'(v)|\, dv = o(\delta^n), \tag{3.33}$$

But now by interpolation (see for instance [10]), using (3.32) and (3.33), we get for some $C > 0$,

$$\begin{aligned} \|h_1 - h_1^\delta\|_{W^{n,1}} &\leq C\sqrt{\|h_1 - h_1^\delta\|_{W^{2n,1}} \|h_1 - h_1^\delta\|_{L^1}} \\ &= \sqrt{o(\delta^{-n}\delta^n)} = o(1). \end{aligned}$$

Hence, since $S \le n$, $\|h_1^\delta\|_{W^{S,1}}$ is bounded independently of δ. Thus, if we define, similarly as in (3.26) an initial condition

$$\tilde{g}^\delta(0) := \mu + \frac{\delta}{2}(h_1^\delta + \overline{h_1^\delta}),$$

then we have ensured that is non-negative, and that $\|\tilde{g}^\delta(0) - \mu\|_{W^{S,1}} \le C\delta$. We denote $\tilde{g}^\delta$ the solution to (1.1) with initial condition $\tilde{g}^\delta(0)$. To compare $\tilde{g}^\delta$ to the previous approximation $\tilde{g}^N_{app}$, we can still apply (3.19) if we add a term for the difference at initial time that does not vanish anymore:

$$\|\tilde{g}^\delta(t) - \tilde{g}^N_{app}(t)\|_1 \le \int_0^t e^{(t-s)(\|\partial_v\mu\|_1+1)}\|\tilde{R}^N_{app}(s)\|_1\,ds + e^{t(\|\partial_v\mu\|_1+1)}\|\tilde{g}^\delta(0) - \tilde{g}(0)\|_1,$$

and the previous analysis can still be done since by (3.33), as $N+1 \le n$, we have

$$\|\tilde{g}^\delta(0) - \tilde{g}(0)\|_{L^1} = o(\delta^{N+1}).$$

This concludes the proof.

3.5 Proof of Proposition 2.1.

First remark that the δ'-condition is weaker than the δ-condition, since the later implies that $V_\delta = W_\delta = \emptyset$, for δ small enough, so that $\int_{W_\delta}|\mu'(v)|\,dv = 0$ and condition (3.31) clearly holds.

• *Point i. implies the δ'-condition.*

First, the positivity of μ implies that for $R > 0$ large enough, $V_\delta \cap [-R,R] = \emptyset$, and also $W_\delta \cap [-R,R] = \emptyset$. Then, the upper bound in (2.6) implies that for some constant $c > 0$

$$V_\delta \subset W_\delta \subset \Big\{v,\ \text{s.t. } |v| \ge v_\delta\Big\}, \qquad \text{with} \qquad v_\delta := c\,\delta^{-\frac{1}{\alpha-1}}.$$

Next, remark that the lower bound in (2.6) and the smoothness of μ forbids μ' to change its sign for $|v|$ large enough, so that, μ' is necessarily negative for large v. Then, a straightforward integration of the lower bound in (2.6), leads for $|v|$ large enough to the inequality

$$\mu(v) \le Ce^{-\frac{1}{C\alpha}|v|^{\alpha+1}}.$$

But now

$$\begin{aligned}\int_{W_\delta}|\mu'(v)|\,dv &\le \int_{-\infty}^{-v_\delta}|\mu'(v)|\,dv + \int_{v_\delta}^{+\infty}|\mu'(v)|\,dv = |\mu(v_\delta)| + |\mu(-v_\delta)|\\ &\le 2Ce^{-\frac{1}{C\alpha}|v_\delta|^{\alpha+1}} \le 2Ce^{-c'\delta^{-\gamma}} \quad \text{with} \quad \gamma := \frac{\alpha+1}{\alpha-1}\end{aligned}$$

and the quantity in the last r.h.s. is a $o(\delta^n)$ for any $n \in \mathbb{N}$.

• *Point ii. implies the δ'-condition.*

Using the positivity of μ on the interior of the (a_i, b_i), the upper bound in (2.7) and arguing similarly to the previous step, we obtain that for some constants $c_i, c_i' > 0$

$$V_\delta \subset \bigcup_i (a_i, a_i^\delta) \,\cup\, \bigcup_i (b_i, b_i^\delta), \quad \text{where} \quad a_i^\delta := a_i + c_i\delta^{\frac{1}{\beta}}, \quad b_i^\delta := b_i - c_i'\delta^{\frac{1}{\beta}}.$$

It then implies with $\beta' := \max(\beta, 2)$ that for some different constants $c_i, c_i' > 0$,

$$W_\delta \subset \bigcup_i (a_i, \tilde{a}_i^\delta) \,\cup\, \bigcup_i (\tilde{b}_i^\delta, b_i^\delta), \quad \text{where} \quad \tilde{a}_i^\delta := a_i + c_i \delta^{\frac{1}{\beta'}}, \quad \tilde{b}_i^\delta := b_i - c_i' \delta^{\frac{1}{\beta'}}.$$

Moreover, the lower bound in (2.7) implies that μ' does not change its sign closely above a_i (and also closely below b_i). Then,

$$\begin{aligned}\int_{W_\delta} |\mu'(v)|\, dv &\le \sum_i \int_{a_i}^{\tilde{a}_i^\delta} |\mu'(v)|\, dv + \sum_i \int_{\tilde{b}_i^\delta}^{b_i} |\mu'(v)|\, dv \\ &\le \sum_i \mu\big(\tilde{a}_i^\delta\big) + \sum_i \mu\big(\tilde{b}_i^\delta\big).\end{aligned}$$

But since μ is smooth and vanishes at any order at a_i and b_i, we have that $\mu\big(\tilde{a}_i^\delta\big) = o(\delta^n)$ for any $n \in \mathbb{N}$ (we have as well a similar behavior for the $\tilde{b}_i^\delta$). This implies condition (3.31).

• *The case where μ has zero inside its support.*

We keep the notation of the previous step, but assume now that a_1 is a zero of finite order m of μ. Then the ratio $\frac{\mu'}{(1+|v|)\mu(v)}$ behaves like $\frac{c}{(v-a_1)}$, and from the definition of V_δ we see that for some constant $c_1 > 0$

$$(a_1, a_1^\delta) \subset V_\delta \subset W_\delta, \quad \text{with } a_1^\delta := a_1 + c_1 \delta^{-1}.$$

Then, we have

$$\int_{W_\delta} |\mu'(v)|\, dv \ge \int_{a_1}^{a_1^\delta} |\mu'(v)|\, dv = \mu(a_1^\delta) \sim c\delta^m,$$

for some $c > 0$, so that the condition (3.31) is not satisfied for $n > m$.

4 Stable case: proof of Theorems 2.2 and 2.3

4.1 Some properties of the Casimir functional H_Q.

We gather in the following Proposition some useful properties of the Casimir functional.

Proposition 4.1. *Let μ be a S-stable profile (See definition 2.2) associated by* (2.8) *to a profile φ. Let H_Q be an associated Casimir functional (See Definition 2.3) defined thanks to an admissible function Q. Then :*

i) In general, the quantity $H_Q(f)$ is well defined in $\mathbb{R}^+ \cup \{+\infty\}$ as the integral of a nonnegative measurable function. But if in addition, f satisfies

$$\int f(1 + v^{2+\eta})\, dxdv < +\infty, \quad \textit{for some } \eta > 0, \tag{4.1}$$

then the integrals $\int Q(\mu)$, $\int Q'(\mu) f$ are finite and $\int Q(f)$ is bounded from below.

The same also holds if the profile μ satisfies

$$\int \mu(1 + v^{2+\eta'}) dv < +\infty, \quad \textit{for some } \eta' > 0, \tag{4.2}$$

and $f \in L^1$ and has finite kinetic energy.

ii) *H_Q is convex and non-negative, lower semi-continuous on the space of functions $f \in L^1$ with finite kinetic energy. It vanishes only for $f = \mu$. If moreover Q is uniformly convex: $Q'' \geq \alpha$, for some $\alpha > 0$, then H_Q control the L^2 norm*

$$\|f - \mu\|_2^2 \leq \frac{1}{\alpha} H_Q(f).$$

iii) *When μ is a Maxwellian, $\mu(v) = \frac{1}{\sqrt{2\pi T}} e^{-\frac{|v-u|^2}{2T}}$ for some $T > 0$ and $u \in \mathbb{R}$, then the choice $Q(s) = T\, s \ln s$ is admissible (up to a constant) and the associated Casimir functional H_Q is just the usual relative entropy:*

$$H_Q(f) := H(f|\mu) = \int_{\mathbb{R}} \ln \frac{f}{\mu} f\, dx.$$

Moreover, the Csiszár-Kullback-Pinsker inequality implies that, for all $f \in L^1_{x,v}$, with $f \geq 0, \int f = 1$, we have

$$\|f - \mu\|^2_{L^1_{x,v}} \leq \frac{1}{2} H_Q(f).$$

iv) *If a sequence μ_n of S-stable profile converges weakly towards a Dirac mass $\delta_{\bar{v}}$, for some $\bar{v}$ in $\mathbb{R}$, in such a way that*

$$\int_{\mathbb{R}} |v - \bar{v}|^2 \mu_n(v)\, dv \xrightarrow[n \to +\infty]{} 0, \tag{4.3}$$

then for any compatible sequence H_n of functionals, we have for any bounded $f \geq 0$ satisfying (4.1)

$$H_n(f) \xrightarrow[n \to +\infty]{} \frac{1}{2} \int_{\mathbb{R} \times \mathbb{T}} f(x, v) |v - \bar{v}|^2\, dv dx.$$

Remark 4.1. *The last point is interesting because it tells us that the natural extension of the Casimir functional in the case where the profile is a Dirac mass, is $\frac{1}{2} \int |v - \bar{v}|^2 f\, dv dx$. This is exactly the quantity that is introduced when a modulated energy method is used in the zero temperature limit [16, 44, 35].*

Proof of Proposition 4.1.

Proof of the point i). We shall first prove that condition (2.8) and the monotonicity of φ imply that $\varphi(u)|u|^{\frac{3}{2}} \leq C$ for some $C \in \mathbb{R}^+$. In fact, since φ is increasing, we have the following inequality

$$\sum_{n=0}^{\infty} 2^{\frac{3n}{2}} \varphi(-2^{n+1}) = \sum_{n=0}^{\infty} (2^{n+1} - 2^n) 2^{\frac{n}{2}} \varphi(-2^{n+1}) \leq \int_{-\infty}^{0} \sqrt{-u} \varphi(u)\, du < +\infty.$$

It implies the boundedness of the sequence $2^{\frac{3n}{2}} \varphi(-2^{n+1})$, from which we conclude using monotonicity, that there exists a constant C such that

$$\forall u \in \mathbb{R}^-, \quad 0 < \varphi(u) \leq C(1 - u)^{-\frac{3}{2}}. \tag{4.4}$$

Denoting $a = \varphi(0)$ and using the requirements on Q (See Definition 2.3), this implies that

$$\forall z \in (0, a], \qquad -C^{\frac{2}{3}} z^{-\frac{2}{3}} \leq \varphi^{-1}(z) < 0, \quad \text{and} \quad -\frac{1}{3} C^{\frac{2}{3}} z^{\frac{1}{3}} \leq Q(z) \leq 0. \tag{4.5}$$

Next, the term $\int Q'(\mu)\mu\, dxdv$ which appears in the definition of $\mathcal{L}_\varepsilon$ may be rewritten using that

$$Q'[\mu(v)] = \varphi^{-1} \circ \varphi\Big(-\frac{|v-\bar{v}|^2}{2}\Big) = -\frac{|v-\bar{v}|^2}{2}. \tag{4.6}$$

Thus, it comes

$$\int Q'(\mu)\mu\, dx\, dv = -\frac{1}{2}\int |v-\bar{v}|^2 \varphi\Big(-\frac{|v-\bar{v}|^2}{2}\Big)\, dv,$$

which is finite under condition (2.8). Similarly, we see that the term $\int Q'(\mu) f\, dxdv$ is finite when $f \in L^1$ has finite kinetic energy.

Next, the term $\int Q(\mu)\, dxdv$ is also finite. To see this, use that $Q' = \varphi^{-1}$ on the range of φ, the assumption $Q(0) = 0$ to write for all v

$$Q(\mu(v)) = H(u) := \int_0^{\varphi(u)} \varphi^{-1}(r)\, dr, \quad \text{with } u := -\frac{|v-\bar{v}|^2}{2}.$$

Here we can use the following relation, which is clear from inspection of the graph of φ (or a formal differentiation)

$$H(u) = \int_0^{\varphi(u)} \varphi^{-1}(r)\, dr = u\varphi(u) - \int_{-\infty}^{u} \varphi(s)\, ds.$$

It leads to

$$\begin{aligned}
\int_{\mathbb{T}\times\mathbb{R}} Q(\mu(v))\, dxdv &= \int_{-\infty}^{0} H(u) \frac{du}{\sqrt{-u}} \\
&= -\int_{-\infty}^{0} \sqrt{-u}\, \varphi(u)\, du - \int_{-\infty}^{0} \Big(\int_{-\infty}^{u} \varphi(s)\, ds\Big) \frac{du}{\sqrt{-u}} \\
&= -\int_{-\infty}^{0} \sqrt{-u}\, \varphi(u)\, du - \int_{-\infty}^{0} \Big(\int_{s}^{0} \frac{du}{\sqrt{-u}}\Big) \varphi(s)\, ds \\
&= -3\int_{-\infty}^{0} \sqrt{-u}\, \varphi(u)\, du,
\end{aligned} \tag{4.7}$$

which is finite by assumption (2.8).

Finally, using the bound by below (4.5) for Q on $[0,a]$ and the simpler bound $Q(z) \geq -bz - c$ for $z \geq a$ (such nonnegative constants b and c exist since Q is convex), we get

$$\begin{aligned}
\int_{\mathbb{T}\times\mathbb{R}} Q(f)\, dx\, dv &= \int_{\{f\leq a\}} Q(f)\, dx\, dv + \int_{\{f\geq a\}} Q(f)\, dx\, dv \\
&\geq -C\int f^{\frac{1}{3}}\, dx\, dv - \int_{\{f\geq a\}} (bf + c)\, dx\, dv \\
&\geq -C\int \big[f(1+v^{2+\eta})\big]^{\frac{1}{3}} \frac{dx\, dv}{(1+v^{2+\eta})^{\frac{1}{3}}} - b\|f\|_1 - c\int_{\{f\geq a\}} dx\, dv \\
&\geq -C\Big[\int f(1+v^{2+\eta})\, dx\, dv\Big]^{\frac{1}{3}} \Big[\int (1+v^{2+\eta})^{-\frac{1}{2}}\, dx\, dv\Big]^{\frac{2}{3}} - \Big(b + \frac{c}{a}\Big)\|f\|_1 \\
&\geq -C_\eta\Big[\int f(1+v^{2+\eta})\, dx\, dv\Big]^{\frac{1}{3}} - \Big(b + \frac{c}{a}\Big)\|f\|_1,
\end{aligned} \tag{4.8}$$

where we have used on the fourth line the Hölder inequality.

The case where μ satisfies (4.2) allows to improve the bound from below (4.5) by

$$\forall z \in (0, a], \qquad -C' z^{\frac{1+\eta}{3+\eta}} \le Q(z) \le 0.$$

Having seen this, the same calculations can be done using only that f has finite kinetic energy.

All in all, we see that all the integrals composing H_Q are well defined: the first one belongs to $\mathbb{R} \cup \{+\infty\}$ and the three other ones are finite.

Proof of Points ii) and iii). The convexity and nonnegativity of H_Q are clear. The convexity of Q implies that the first term in the definition (2.9) of H_Q is l.s.c.. Two others are constant, and the last one may be rewritten $\int |v - \bar{v}|^2 f(x, v)\, dx\, dv$ which is l.s.c on the space of functions $f \in L^1$ with finite kinetic energy. The fact that H_Q vanishes only at μ is a consequence of the uniform convexity of Q on the support of φ^{-1} (that is also the range of μ).

The point $iii)$ is a simple consequence of a short calculation that we skip. The L^1 control is the classical Csiszár-Kullback-Pinsker inequality.

Proof of Point iv). Under the assumption (4.3), the short argument at the beginning of the proof of Point $i)$ implies that the bounds (4.5) hold for each n, with a constant C_n that goes to zero as $n \to +\infty$. Remark also that $a_n = \varphi_n(0) \to +\infty$.

If $\|f\|_\infty < +\infty$, we will have for n large enough $f(x, v) \le a_n$ for all (x, v), and this implies that $\int Q_n(f) \le 0$. It is also not difficult to see that the bound by below obtained in (4.8) goes to zero has n goes to infinity. Thus

$$\lim_{n \to +\infty} \int_{\mathbb{T}\times\mathbb{R}} Q_n(f(x, v))\, dxdv = 0, \qquad \lim_{n \to +\infty} \int_{\mathbb{T}\times\mathbb{R}} Q_n(\mu_n(x, v))\, dxdv = 0,$$

thanks to (4.7) and the assumption (4.3). The term $\int Q_n'(\mu_n)$ vanishes also in the limit because it is exactly the term that appears in (4.3). The last remaining term is constant and is equal to $\frac{1}{2} \int |v - \bar{v}|^2 f\, dvdx$, and this concludes the proof. □

4.2 The well prepared case.

In this paragraph we prove Theorem 2.2. Recall that

$$\mathcal{L}_\varepsilon(t) = \frac{\varepsilon^2}{2} \int |\partial_x V_\varepsilon(t)|^2 dx + \int \big[Q(f_\varepsilon(t)) - Q(\mu) - Q'(\mu)(f_\varepsilon(t) - \mu)\big]\, dvdx. \tag{4.9}$$

Since our solutions f_ε are strong ones, the term $\int Q(f_\varepsilon(t))\, dxdv$ is exactly independent of the time:

$$\int Q(f_\varepsilon(t))\, dxdv = \int Q(f_{\varepsilon,0})\, dxdv.$$

We introduce the current j_ε, defined as follows:

$$j_\varepsilon(t, x) := \int f_\varepsilon(t, x, v) v\, dv.$$

Since the two other constants term $\int Q(\mu)\, dxdv$ and $\int Q'(\mu)\mu\, dxdv$ are finite, it remains to understand how behave

$$\begin{aligned}
\frac{\varepsilon}{2} \int |\partial_x V_\varepsilon(t)|^2 dx - \int Q'(\mu) f_\varepsilon(t) dvdx &= \frac{\varepsilon}{2} \int |\partial_x V_\varepsilon(t)|^2 dx + \frac{1}{2} \int |v - \bar{v}|^2 f_\varepsilon(t) dvdx. \\
&= \mathcal{E}_\varepsilon[f_\varepsilon] - \bar{v} \int j_\varepsilon\, dx + \frac{|\bar{v}|^2}{2} \int \rho_\varepsilon\, dx,
\end{aligned}$$

where we have used (4.6). Since the total mass $\int \rho_\varepsilon \, dx$, and the total momentum $\int j_\varepsilon \, dx$, and the total energy $\mathcal{E}_\varepsilon[f_\varepsilon]$ are preserved by strong solutions of the Vlasov-Poisson equation (1.1), we finally conclude that $\mathcal{L}_\varepsilon$ is constant.

4.3 Plasma oscillations.

The above analysis is only useful in the well-prepared case, that is when the potential energy vanishes in the limit: $\|\partial_x V_{0,\varepsilon}\|_2 = o(\varepsilon^{-1})$. In general, as already evoked in the introduction, there are time oscillations of the electric field, called plasma oscillations, that we have to take into account. In this paragraph, we give a description of these, with the aim to explain the form of the filtered functionals of Theorem 2.3.

For any $\varepsilon > 0$, we consider f_ε a solution to (1.1). Then the density ρ_ε and the current j_ε satisfy the system of equations

$$\begin{cases} \partial_t \rho_\varepsilon + \partial_x j_\varepsilon = 0, \\ \partial_t j_\varepsilon + \partial_x \Big(\displaystyle\int f_\varepsilon v^2 \, dv \Big) + \partial_x V_\varepsilon \rho_\varepsilon = 0. \end{cases} \tag{4.10}$$

Fast oscillations are hidden in that system. Since we work in dimension one, we can always write

$$j_\varepsilon(t,x) = \bar{j}_\varepsilon(t) + \partial_x J_\varepsilon(t,x), \tag{4.11}$$

which is the analogue of the decomposition of a vector-field in potential part and divergence free part in higher dimension. The so-called "potential of the current" J_ε is defined up to a constant, that we can choose later. Using the Poisson equation of (1.1) in the two lines of (4.10), we get

$$\begin{cases} -\varepsilon^2 \partial_{txx} V_\varepsilon + \partial_{xx} J_\varepsilon = 0, \\ \partial_t \bar{j}_\varepsilon + \partial_{tx} J_\varepsilon + \partial_x \Big(\displaystyle\int f_\varepsilon v^2 \, dv \Big) + \partial_x V_\varepsilon - \dfrac{\varepsilon^2}{2} \partial_x |\partial_x V_\varepsilon|^2 = 0. \end{cases}$$

In the second line, the gradient part in x and the constant part can be solved separately thanks to the periodicity, and we first get that $\bar{j}_\varepsilon(t) = \bar{j}_{0,\varepsilon}$ for all times t. Then the equation on $\partial_x J_\varepsilon$ and $\partial_x V_\varepsilon$ can be rewritten as

$$\begin{cases} \partial_t(\varepsilon \, \partial_x V_\varepsilon) & = \dfrac{\partial_x J_\varepsilon}{\varepsilon}, \\ \partial_t(\partial_x J_\varepsilon) & = -\dfrac{\varepsilon \, \partial_x V_\varepsilon}{\varepsilon} + \partial_x \Big(\dfrac{1}{2} |\varepsilon \, \partial_x V_\varepsilon|^2 - \displaystyle\int f_\varepsilon v^2 \, dv \Big), \end{cases} \tag{4.12}$$

which may also rewritten directly on V_ε and J_ε, which are defined only up to some constant. Thanks to that, we can define a complex quantity

$$\mathcal{O}_\varepsilon(t,x) := J_\varepsilon(t,x) + i \, \varepsilon \, V_\varepsilon(t,x),$$

which satisfies

$$\partial_t \mathcal{O}_\varepsilon = \frac{i}{\varepsilon} \mathcal{O}_\varepsilon + \frac{1}{2} |\partial_x (\operatorname{Im} \mathcal{O}_\varepsilon)|^2 - \int f_\varepsilon v^2 \, dv.$$

Using Duhamel formula, we obtain

$$e^{-i \frac{t}{\varepsilon}} \mathcal{O}_\varepsilon(t) = \mathcal{O}_{0,\varepsilon} + \int_0^t e^{-i \frac{s}{\varepsilon}} \left(\frac{1}{2} |\partial_x (\operatorname{Im} \mathcal{O}_\varepsilon)|^2 - \int f_\varepsilon v^2 \, dv \right) ds. \tag{4.13}$$

This means that there are large oscillations of period ε in $\partial_x V_\varepsilon$ and $\partial_x J_\varepsilon$, respectively of amplitude or order ε^{-1} and 1. For this reason, even if $f_{0,\varepsilon}$ converges to a stable equilibrium $\mu(v)$, we can not expect f_ε to converge to μ, without filtrating these oscillations. What we expect is something like $f_\varepsilon(t,x,v-\partial_x J_\varepsilon) \approx f(v)$. In particular, $\int f_\varepsilon v\,dv$ and $\int f_\varepsilon(v)v^2\,dv$ should not converge to $\int f v\,dv$ and $\int f(v)v^2\,dv$, but we expect something like

$$\begin{aligned}
&\int f_\varepsilon(t,x,v)\,dv = \rho_\varepsilon \approx \int \mu(v)\,dv = 1,\\
&\int f_\varepsilon(t,x,v)v\,dv = j_\varepsilon(t,x) \approx \bar v + \partial_x J_\varepsilon(t,x),\\
&\int f_\varepsilon(t,x,v)v^2\,dv \approx \int \mu(v)|v+\partial_x J_\varepsilon(t,x)|^2\,dv = 2T + |\bar v + \partial_x J_\varepsilon(t,x)|^2,
\end{aligned}$$

with the notation $\bar v := \int \mu(v)v\,dv$ and $T := \frac{1}{2}\int \mu(v)|v-\bar v|^2\,dv$. Using this in equation (4.13), we get, if we forget the constants which are not important at the level of potentials,

$$e^{-i\frac{t}{\varepsilon}}\mathcal{O}_\varepsilon(t) = \mathcal{O}_{0,\varepsilon} + \int_0^t e^{-i\frac{s}{\varepsilon}}\left(\frac{1}{2}|\partial_x(\operatorname{Im}\mathcal{O}_\varepsilon)|^2 - |\partial_x(\operatorname{Re}\mathcal{O}_\varepsilon)+\bar v|^2\right)ds. \tag{4.14}$$

The term $\mathcal{O}_\varepsilon$ displays fast oscillations in time, but we can try to rewrite everything in terms of

$$\mathcal{U}_\varepsilon(t,x) := e^{-i\frac{t}{\varepsilon}}\mathcal{O}_\varepsilon(t,x),$$

which has a bounded derivative in time, because both the kinetic and potential energy are bounded. We need to write $|\operatorname{Im}\partial_x\mathcal{O}_\varepsilon|^2$ and $|\operatorname{Re}\partial_x\mathcal{O}_\varepsilon|^2$ in terms of $\partial_x\mathcal{U}_\varepsilon$. It comes

$$\begin{aligned}
|\operatorname{Im}\partial_x\mathcal{O}_\varepsilon|^2 &= \frac{1}{4}|\partial_x\mathcal{O}_\varepsilon - \overline{\partial_x\mathcal{O}_\varepsilon}|^2 = \frac{1}{4}\left|\partial_x[e^{-i\frac{t}{\varepsilon}}\mathcal{O}_\varepsilon] - e^{-2i\frac{t}{\varepsilon}}\overline{\partial_x[e^{-i\frac{t}{\varepsilon}}\mathcal{O}_\varepsilon]}\right|^2\\
&= \frac{1}{4}\left|\partial_x\mathcal{U}_\varepsilon - e^{-2i\frac{t}{\varepsilon}}\overline{\partial_x\mathcal{U}_\varepsilon}\right|^2,\\
e^{-i\frac{s}{\varepsilon}}|\operatorname{Im}\partial_x\mathcal{O}_\varepsilon(s)|^2 &= \frac{e^{-i\frac{s}{\varepsilon}}}{2}|\partial_x\mathcal{U}_\varepsilon|^2 - \frac{1}{4}\partial_x\mathcal{U}_\varepsilon\overline{\partial_x\mathcal{U}_\varepsilon}\left(e^{i\frac{s}{\varepsilon}} + e^{-3i\frac{s}{\varepsilon}}\right).
\end{aligned}$$

So since $\mathcal{U}_\varepsilon$ contains no fast oscillations in time, $e^{-i\frac{s}{\varepsilon}}|\operatorname{Im}\partial_x\mathcal{O}_\varepsilon(s)|^2$ is a sum of terms with fast oscillations in time. So we can expect its contribution in (4.14) to be small and therefore we neglect it. The same holds for $e^{-i\frac{s}{\varepsilon}}|\operatorname{Re}\partial_x\mathcal{O}_\varepsilon(s)|^2$. The only non constant term that will contribute is in fact the one coming from $e^{-i\frac{s}{\varepsilon}}\bar v\operatorname{Re}\partial_x\mathcal{O}_\varepsilon(s)$, since

$$\begin{aligned}
\operatorname{Re}\partial_x\mathcal{O}_\varepsilon(s) &= \frac{1}{2}[\partial_x\mathcal{O}_\varepsilon(s) + \overline{\partial_x\mathcal{O}_\varepsilon(s)}] = \frac{1}{2}\left[e^{i\frac{s}{\varepsilon}}\partial_x\mathcal{U}_\varepsilon(s) + e^{-i\frac{s}{\varepsilon}}\overline{\partial_x\mathcal{U}_\varepsilon(s)}\right],\\
2\bar v e^{-i\frac{s}{\varepsilon}}\operatorname{Re}\partial_x\mathcal{O}_\varepsilon(s) &= \bar v\,\partial_x\mathcal{U}_\varepsilon(s) + \bar v e^{-2i\frac{s}{\varepsilon}}\overline{\partial_x\mathcal{U}_\varepsilon(s)}.
\end{aligned}$$

Only the first term in the r.h.s. of the last equation will have a significant contribution, and if all the approximations are justified, we will end up with a function $\mathcal{U}_\varepsilon$ satisfying approximately the equation

$$\mathcal{U}_\varepsilon(t) = \mathcal{U}_{0,\varepsilon} - \bar v\int_0^t \partial_x\mathcal{U}_\varepsilon(s)\,ds,$$

or equivalently the linear transport equation

$$\partial_t\mathcal{U}_\varepsilon + \bar v\,\partial_x\mathcal{U}_\varepsilon = 0,$$

with the initial condition $\mathcal{U}_{0,\varepsilon} = J_{0,\varepsilon} + i\,\varepsilon V_{0,\varepsilon}$.

If $f_{0,\varepsilon}$ converges towards $\mu(v)$, then the "potential" part $J_{0,\varepsilon}$ of the current converges towards 0. If moreover the potential $\varepsilon\,\partial_x V_{0,\varepsilon}$ has a limit denoted $\partial_x V_0$ as ε goes to zero, then it is natural to define $\mathcal{U}$ as the solution of the simple linear transport equation

$$\begin{cases} \partial_t \mathcal{U} + \bar{v}\,\partial_x \mathcal{U} = 0, \\ \mathcal{U}(0) = iV_0 \end{cases} . \tag{4.15}$$

The solution of (4.15) is simply given by $\mathcal{U}(t,x) := i\,V_0(x-\bar{v}t)$. Therefore, we can use the approximation

$$\mathcal{O}_\varepsilon(t) \approx i\,e^{i\frac{t}{\varepsilon}} V_0(x-\bar{v}t) = V_0(x-\bar{v}t)\left[-\sin\frac{t}{\varepsilon} + i\,\cos\frac{t}{\varepsilon}\right], \tag{4.16}$$

in order to filtrate the plasma oscillations for small ε, hence the expressions of the functionals in Theorem 2.3.

4.4 The general case.

We now prove Theorem 2.3. By Galilean invariance of the Vlasov equation (1.1), we can restrict ourselves to the case $\bar{v}=0$. Indeed, if $\bar{v}\neq 0$, we can rewrite the problem in the variable (x',v') defined by

$$v = \bar{v} + v', \qquad x = x' + \bar{v}t.$$

This will simplify the calculations.

Next with the help of some simple changes of variables, remark that we may rewrite

$$H_Q\left[f_\varepsilon\left(t,x,v-\partial_x V_0 \sin\frac{t}{\varepsilon}\right)\right] = \int Q(f_\varepsilon)\,dx\,dv + \frac{1}{2}\int\left|v+\partial_x V_0\sin\frac{t}{\varepsilon}\right|^2 f_\varepsilon(t,x,v)\,dx\,dv - \int Q(\mu)\,dv - \int \frac{v^2}{2}\mu(v)\,dv \tag{4.17}$$

Since, we are dealing with strong solution, the first term in the r.h.s. of (4.17) is constant. In view of Proposition 4.1, the last two are also finite constants.

We may also develop the kinetic energy term, which yields

$$\frac{1}{2}\int\left|v+\partial_x V_0\sin\frac{t}{\varepsilon}\right|^2 f_\varepsilon(t,x,v)\,dxdv = \frac{1}{2}\int v^2 f_\varepsilon(t)\,dx\,dv + \frac{1}{2}\sin^2\frac{t}{\varepsilon}\int |\partial_x V_0|^2 \rho_\varepsilon(t,x)\,dx + \sin\frac{t}{\varepsilon}\int \partial_x V_0\, j_\varepsilon(t,x)\,dx. \tag{4.18}$$

Finally, the term with the electric field in $\mathcal{L}_\varepsilon^O$ leads to

$$\frac{1}{2}\int\left|\varepsilon\partial_x V_\varepsilon - \partial_x V_0\cos\frac{t}{\varepsilon}\right|^2 dx = \frac{1}{2}\int |\varepsilon\partial_x V_\varepsilon|^2\,dx + \frac{1}{2}\cos^2\frac{t}{\varepsilon}\int |\partial_x V_0(x)|^2\,dx - \cos\frac{t}{\varepsilon}\int \partial_x V_0\,[\varepsilon\partial_x V_\varepsilon]\,dx. \tag{4.19}$$

Summing up the first terms in the r.h.s in (4.18) and (4.19), we get the total energy $\mathcal{E}_\varepsilon[f_\varepsilon(t)]$, which is preserved by the dynamics. Moreover, up to a constant, we can replace the $\cos^2$ appearing in (4.19) by a $-\sin^2$. Finally, summing up (4.18) and (4.19), we get that

$$\mathcal{L}_\varepsilon^O(t) = \mathcal{K}_\varepsilon^O(t) + \text{a constant term}$$

where

$$\mathcal{K}_\varepsilon^O(t) := \frac{1}{2}\sin^2\frac{t}{\varepsilon}\int |\partial_x V_0|^2(\rho_\varepsilon - 1)\,dx + \int \partial_x V_0\Big[j_\varepsilon \sin\frac{t}{\varepsilon} - \varepsilon\partial_x V_\varepsilon \cos\frac{t}{\varepsilon}\Big]\,dx. \tag{4.20}$$

Using the Poisson equation in (1.1), the decomposition of j_ε in its constant and potential part $j_\varepsilon = \bar{j}_\varepsilon + \partial_x J_\varepsilon$, and introducing

$$\partial_x R_\varepsilon := \partial_x J_\varepsilon \sin\frac{t}{\varepsilon} - \varepsilon\partial_x V_\varepsilon \cos\frac{t}{\varepsilon}, \tag{4.21}$$

we can express $\mathcal{K}_\varepsilon^O$ in terms of $\partial_x R_\varepsilon$ only:

$$\mathcal{K}_\varepsilon^O(t) = -\frac{1}{2}\sin^2\frac{t}{\varepsilon}\int |\partial_x V_0|^2\,\varepsilon^2\partial_{xx}V_\varepsilon\,dx + \int \partial_x V_0\,\partial_x R_\varepsilon\,dx. \tag{4.22}$$

After an integration by parts, we can rewrite it as

$$\begin{aligned}\mathcal{K}_\varepsilon^O(t) &= \varepsilon\sin^2\frac{t}{\varepsilon}\int \partial_{xx}V_0\partial_x V_0\,\varepsilon\partial_x V_\varepsilon\,dx + \int \partial_x V_0\,\partial_x R_\varepsilon\,dx\\ &=: \qquad \mathcal{I}_\varepsilon^{O,1}(t) \qquad + \qquad \mathcal{I}_\varepsilon^{O,2}(t).\end{aligned}$$

It can be shown the first term $\mathcal{I}_\varepsilon^{O,1}$ is of order ε, but we shall not use this fact, because we need this term in order to compensate for some bad terms coming form the time derivative of $\mathcal{I}_\varepsilon^{O,2}(t)$. The interest of $\partial_x R_\varepsilon$ is that thanks to (4.12), we have

$$\partial_t[\partial_x R_\varepsilon] = \partial_x\Big(\frac{1}{2}|\varepsilon\partial_x V_\varepsilon|^2 - \int f_\varepsilon v^2\,dv\Big)\sin\frac{t}{\varepsilon}, \tag{4.23}$$

so that the possible oscillations with amplitude ε^{-1} in $\partial_t\mathcal{K}_\varepsilon^O$ will vanish. Using once again (4.12), it comes

$$\frac{d}{dt}\mathcal{I}_\varepsilon^{O,1}(t) = 2\sin\frac{t}{\varepsilon}\cos\frac{t}{\varepsilon}\int \partial_{xx}V_0\partial_x V_0\,\varepsilon\partial_x V_\varepsilon\,dx + \sin^2\frac{t}{\varepsilon}\int \partial_{xx}V_0\partial_x V_0\partial_x J_\varepsilon\,dx, \tag{4.24}$$

$$\begin{aligned}\frac{d}{dt}\mathcal{I}_\varepsilon^{O,2}(t) &= \sin\frac{t}{\varepsilon}\int \partial_x V_0\partial_x\Big(\frac{1}{2}|\varepsilon\partial_x V_\varepsilon|^2 - \int f_\varepsilon v^2\,dv\Big)\,dx\\ &= -\frac{1}{2}\sin\frac{t}{\varepsilon}\int \partial_{xx}V_0\,|\varepsilon\partial_x V_\varepsilon|^2\,dx + \sin\frac{t}{\varepsilon}\int f_\varepsilon v^2\partial_{xx}V_0\,dvdx.\end{aligned} \tag{4.25}$$

We shall now write the two decompositions

$$\begin{aligned}-\frac{1}{2}\sin\frac{t}{\varepsilon}\int \partial_{xx}V_0|\varepsilon\partial_x V_\varepsilon|^2\,dx &= -\frac{1}{2}\sin\frac{t}{\varepsilon}\int \partial_{xx}V_0\left|\varepsilon\partial_x V_\varepsilon - \partial_x V_0\cos\frac{t}{\varepsilon}\right|^2\,dx\\ &\quad - \sin\frac{t}{\varepsilon}\cos\frac{t}{\varepsilon}\int \partial_{xx}V_0\partial_x V_0\,\varepsilon\partial_x V_\varepsilon\,dx + \frac{1}{2}\cos^2\frac{t}{\varepsilon}\sin\frac{t}{\varepsilon}\int \partial_{xx}V_0|\partial_x V_0|^2\,dx,\end{aligned} \tag{4.26}$$

$$\begin{aligned}\sin\frac{t}{\varepsilon}\int f_\varepsilon v^2\partial_{xx}V_0\,dvdx &= \sin\frac{t}{\varepsilon}\int f_\varepsilon\left|v + \partial_x V_0\sin\frac{t}{\varepsilon}\right|^2\partial_{xx}V_0\,dvdx\\ &\quad - \sin^3\frac{t}{\varepsilon}\int \rho_\varepsilon|\partial_x V_0|^2\partial_{xx}V_0\,dx - 2\sin^2\frac{t}{\varepsilon}\int \partial_{xx}V_0\partial_x V_0\,j_\varepsilon\,dx.\end{aligned} \tag{4.27}$$

Note that in the first decomposition, the last term of the r.h.s. is actually equal to 0 by integration by parts.

The interest of these two decompositions is that it introduces terms which are very similar to those of $\mathcal{L}_\varepsilon^O(t)$: up to the multiplicative factor $\partial_{xx}V_0$, the first term in the r.h.s. of (4.26) is the relative potential energy term of $\mathcal{L}^O_\varepsilon(t)$, and the first term in the r.h.s. of (4.27) appears in the relative entropy part of $\mathcal{L}^O_\varepsilon(t)$ (See (4.17)). The idea is then to let appear the missing terms. To this end, remark that since μ does not depend on x, by integration by parts, we have

$$\int Q(\mu)\,\partial_{xx}V_0\,dvdx = 0, \qquad \int Q'(\mu)\mu\,\partial_{xx}V_0\,dvdx = 0.$$

Thus, using (4.24)–(4.27), we finally end up with

$$\frac{d}{dt}\mathcal{L}^O_\varepsilon(t) = -\sin\frac{t}{\varepsilon}\int\left\{\int 2\big[Q\left(\tilde{f}_\varepsilon\right) - Q(\mu) - Q'(\mu)(\tilde{f}_\varepsilon - \mu)\big]\,dv + \frac{1}{2}\Big[\varepsilon\partial_x V_\varepsilon - \partial_x V_0\cos\frac{t}{\varepsilon}\Big]^2\right\}$$
$$\times\,\partial_{xx}V_0\,dx + \sin\frac{t}{\varepsilon}\,B^1_\varepsilon(t) + \sin\frac{t}{\varepsilon}B^2_\varepsilon(t) + r_\varepsilon(t), \quad (4.28)$$

where $\tilde{f}_\varepsilon(t,x,v) := f_\varepsilon\left(t,x,v-\partial_x V_0\sin\frac{t}{\varepsilon}\right)$ and with:

$$\begin{aligned}
B^1_\varepsilon(t) &:= \int \partial_{xx}V_0\,\partial_x V_0\Big(\cos\frac{t}{\varepsilon}\,\varepsilon\partial_x V_\varepsilon - \sin\frac{t}{\varepsilon}\,\partial_x J_\varepsilon\Big)\,dx = \int \partial_{xx}V_0\,\partial_x V_0\,\partial_x R_\varepsilon\,dx,\\
B^2_\varepsilon(t) &:= 2\int Q(f_\varepsilon)\,\partial_{xx}V_0\,dvdx,\\
r_\varepsilon(t) &:= -\sin^3\frac{t}{\varepsilon}\int \rho_\varepsilon|\partial_x V_0|^2\partial_{xx}V_0\,dx.
\end{aligned}$$

In what follows, we will show that r_ε is of order ε, and that B^1_ε and B^2_ε are bounded with bounded time derivatives, so that the contributions of $\sin\frac{t}{\varepsilon}\,B^1_\varepsilon$ and $\sin\frac{t}{\varepsilon}B^2_\varepsilon$ will be of order ε after integration in time.

Integrating (4.28) in time and using the convexity of Q, we obtain the bound

$$\mathcal{L}^O_\varepsilon(t) \le \mathcal{L}^O_\varepsilon(0) + 2\,\|\partial_{xx}V_0\|_{L^\infty}\int_0^t \mathcal{L}^O_\varepsilon(s),ds + \int_0^t\Big(\sin\frac{s}{\varepsilon}\,B^1_\varepsilon(s) + \sin\frac{s}{\varepsilon}\,B^2_\varepsilon(s) + r_\varepsilon(s)\Big)\,ds. \quad (4.29)$$

• *Treatment of the time integral with B^i_ε for $i=1,2$.* Since B^1_ε and B^2_ε are not small, but have bounded derivatives, their respective contribution in (4.29) can be controlled with the help of an integration by parts. Precisely, we have

$$\begin{aligned}
\int_0^t \sin\frac{s}{\varepsilon}\,B^i_\varepsilon(s)\,ds &= \int_0^t\Big(\varepsilon\cos\frac{s}{\varepsilon}\Big)'\,B^i_\varepsilon(s)\,ds\\
&= -\varepsilon\int_0^t\cos\frac{s}{\varepsilon}B^{i\,\prime}_\varepsilon(s)\,ds + \Big[\cos\frac{s}{\varepsilon}B^i_\varepsilon(s)\Big]_0^t,
\end{aligned}$$

and this leads to

$$\left|\int_0^t \sin\frac{s}{\varepsilon}\,B^i_\varepsilon(s)\,ds\right| \le \varepsilon\,\|B^{i\,\prime}_\varepsilon\|_\infty\,t + 2\varepsilon\,\|B^i_\varepsilon\|_\infty \quad (4.30)$$

So, it only remains to get uniform (in time) bounds on B^i_ε and its derivative for $i=1,2$.

• *Uniform bounds on B^1_ε and $B^{1\,\prime}_\varepsilon$.*

According to the definition of B^1_ε and (4.21), we have the following bound

$$\begin{aligned}
|B^1_\varepsilon(t)| &\le \|\partial_{xx}V_0\|_\infty\,\|\partial_x V_0\|_\infty\,\|\partial_x R_\varepsilon\|_{L^1}\\
&\le \|\partial_{xx}V_0\|^2_\infty\big(\|\varepsilon\partial_x V_\varepsilon\|_{L^1} + \|\partial_x J_\varepsilon\|_{L^1}\big)\\
&\le \|\partial_{xx}V_0\|^2_\infty\big(\|\varepsilon\partial_x V_\varepsilon\|_{L^2} + 2\,\|j_\varepsilon\|_{L^1}\big)\\
&\le \|\partial_{xx}V_0\|^2_\infty\big(1 + \|\varepsilon\partial_x V_\varepsilon\|^2_{L^2} + \|f_\varepsilon\|_{L^1} + \|f_\varepsilon v^2\|_{L^1}\big)\\
&\le 2\,\|\partial_{xx}V_0\|^2_\infty\big(1 + \mathcal{E}_{\varepsilon,0}\big),
\end{aligned}$$

where we used the notation $\mathcal{E}_{\varepsilon,0} := \mathcal{E}_\varepsilon[f_{0,\varepsilon}]$, the fact that $\partial_x J_\varepsilon = j_\varepsilon - \bar{j}_\varepsilon$ ($\bar{j}_\varepsilon$ being the average of j_ε), and a simple interpolation. Remember also that $\|f_\varepsilon\|_{L^1} = 1$ by assumption on the initial datum.

Now, according to (4.23), we have

$$
\begin{aligned}
|\partial_t B^1_\varepsilon(t)| &:= \left| \int \partial_{xx} V_0 \, \partial_x V_0 \, \partial_t \partial_x R_\varepsilon \, dx \right| \\
&= \left| \sin \frac{t}{\varepsilon} \int \partial_x \left(\partial_{xx} V_0 \partial_x V_0 \right) \left(\frac{1}{2} |\varepsilon \partial_x V_\varepsilon|^2 - \int f_\varepsilon v^2 \, dv \right) dx \right| \\
&\leq 4 \, \|\partial_{xxx} V_0\|_{L^\infty} \, \|\partial_{xx} V_0\|_{L^\infty} \, \mathcal{E}_{\varepsilon,0}.
\end{aligned}
$$

Plugging it into (4.30), we get the bound

$$
\left| \int_0^t \sin \frac{s}{\varepsilon} \, R^1_\varepsilon(s) \, ds \right| \leq 4\varepsilon \, \|\partial_{xx} V_0\|_\infty \big(1 + \mathcal{E}_{\varepsilon,0}\big) \Big(\|\partial_{xxx} V_0\|_{L^\infty} \, t + \|\partial_{xx} V_0\|_\infty \Big). \tag{4.31}
$$

• *Uniform bounds on B^2_ε and $B^{2'}_\varepsilon$.* Since our solutions are strong, we simply bound using the notations introduced in (2.15)

$$
\begin{aligned}
|B^2_\varepsilon(t)| &\leq 2 \, \|\partial_{xx} V_0\|_\infty \int |Q(f_\varepsilon)(t)| \, dx \, dv \\
&= 2 \, \|\partial_{xx} V_0\|_\infty \int |Q(f_{\varepsilon,0})(t)| \, dx \, dv \leq 2 \, \|\partial_{xx} V_0\|_\infty \mathcal{Q}_{\varepsilon,0}.
\end{aligned}
$$

To bound the time derivative of B^2_ε, we rely on the fact that $Q(f_\varepsilon)$ is also a strong solution of the Vlasov equation (1.1):

$$
\partial_t Q(f_\varepsilon) + v \, \partial_x \big[Q(f_\varepsilon)\big] - \partial_x V_\varepsilon \partial_v \big[Q(f_\varepsilon)\big] = 0.
$$

We thus get

$$
\begin{aligned}
\int \partial_t \big[Q(f_\varepsilon)\big] \, \partial_{xx} V_0 \, dv \, dx &= \int \partial_{xxx} V_0 \, v \, Q(f_\varepsilon) \, dv dx - \int Q(f_\varepsilon) \, \partial_v \big[\partial_x V_\varepsilon \, \partial_{xx} V_0\big] \, dv \, dx \\
&= \int \partial_{xxx} V_0 \, v \, \sqrt{f_\varepsilon} \, \frac{Q(f_\varepsilon)}{\sqrt{f_\varepsilon}} \, dv dx.
\end{aligned}
$$

Note that all the underlying calculations are well justified since $Q(f_\varepsilon)$ is assumed to be in $L^1_{x,v}$ and $\partial_{xx} V_0$ and $\partial_x V_\varepsilon$ are bounded (non uniformly in ε). Then

$$
\begin{aligned}
\big|\partial_t B^2_\varepsilon\big| &= \left| \int \partial_t Q(f_\varepsilon) \, \partial_{xx} V_0 \, dv dx \right| \\
&\leq \frac{1}{2} \|\partial_{xxx} V_0\|_\infty \left(\int v^2 f_\varepsilon \, dv dx + \int \frac{Q^2(f_\varepsilon)}{f_\varepsilon} \, dv dx \right) \\
&\leq \|\partial_{xxx} V_0\|_\infty \Big(\mathcal{E}_{\varepsilon,0} + \frac{1}{2} \mathcal{Q}_{\varepsilon,0} \Big).
\end{aligned}
$$

Plugging all into (4.30), we get the bound

$$
\left| \int_0^t \sin \frac{s}{\varepsilon} \, R^1_\varepsilon(s) \, ds \right| \leq \varepsilon \Big[\|\partial_{xxx} V_0\|_\infty \big(\mathcal{E}_{\varepsilon,0} + \mathcal{Q}_{\varepsilon,0}\big) \, t + 4 \|\partial_{xx} V_0\|_\infty \mathcal{Q}_{\varepsilon,0} \Big]. \tag{4.32}
$$

• *Treatment of r_ε.* The last term of the remainder is the easiest to analyze. We use the Poisson equation in (1.1) to write

$$\begin{aligned} r_\varepsilon(t) = & -\sin^3\frac{t}{\varepsilon}\int |\partial_x V_0|^2 \partial_{xx}V_0\,dx + \sin^3\frac{t}{\varepsilon}\int \varepsilon^2\partial_{xx}^2 V_\varepsilon |\partial_x V_0|^2\partial_{xx}V_0\,dx \\ = & -\varepsilon\sin^3\frac{t}{\varepsilon}\int \varepsilon\partial_x V_\varepsilon\,\partial_x\big(|\partial_x V_0|^2\partial_{xx}V_0\big)\,dx \end{aligned}$$

By Cauchy-Schwarz inequality and by conservation of the energy, we deduce that

$$|r_\varepsilon(t)| \le 3\,\varepsilon\,\|\partial_{xx}V_0\|_\infty^2\,\|\partial_{xxx}V_0\|_\infty\big(1+\mathcal{E}_{\varepsilon,0}\big),$$

and a time integration leads to

$$\left|\int_0^t r_\varepsilon(s)\,ds\right| \le 3\,\varepsilon\,\|\partial_{xx}V_0\|_\infty^2\,\|\partial_{xxx}V_0\|_\infty\big(1+\mathcal{E}_{\varepsilon,0}\big)\,t \tag{4.33}$$

• *Conclusion.* For simplicity, we will denote $\kappa := 2\,\|\partial_{xx}V_0\|_{L^\infty}$. Using now (4.29) and gathering the contributions (4.31)–(4.33) together, we have proved that

$$\mathcal{L}_\varepsilon^O(t) \le \mathcal{L}_\varepsilon^O(0) + \kappa\int_0^t \mathcal{L}_\varepsilon^O(s),ds + \varepsilon(a+bt), \tag{4.34}$$

where

$$a := C\,\kappa\big[\kappa(1+\mathcal{E}_{\varepsilon,0}) + \bar{\mathcal{Q}}\big] \quad \text{and} \quad b := C\,\|\partial_{xxx}V_0\|_\infty\big[(1+\kappa^2)(1+\mathcal{E}_{\varepsilon,0}) + \mathcal{Q}_{\varepsilon,0}\big],$$

for some numerical constant $C > 0$. An application of Gronwall lemma leads to the inequality

$$\mathcal{L}_\varepsilon^O(t) \le e^{\kappa t}\Big[\mathcal{L}_\varepsilon^O(0) + \varepsilon\,a + \varepsilon\,\frac{b}{\kappa}\Big],$$

and this concludes the proof of Theorem 2.3, with precisely

$$K := C\big(1+\|\partial_{xx}V_0\|_\infty^2\big)\Big(1+\frac{\|\partial_{xxx}V_0\|_\infty}{\|\partial_{xx}V_0\|_\infty}\Big). \tag{4.35}$$

• *The interpolation argument for Corollary 2.3.* If V_0 has no bounded third derivative, we introduce some smoothing of V_0

$$V_{0,\eta} = V_0 * k_\eta, \qquad k_\eta := \frac{1}{\eta}k\Big(\frac{\cdot}{\eta}\Big),$$

where k is some nonnegative smooth function with compact support and $\int k = 1$. The mollified potential $V_{0,\eta}$ satisfies the following properties:

$$\|\partial_{xx}V_{0,\eta}\|_\infty \le \|\partial_{xx}V_0\|_\infty, \quad \|\partial_{xxx}V_{0,\eta}\|_\infty \le \frac{C}{\eta}\,\|\partial_{xx}V_0\|_\infty, \quad \big\|\partial_x(V_{0,\eta}-V_0)\big\|_2 \le C\,\eta\,\|\partial_{xx}V_0\|_\infty$$

for some constants C depending only on k. It implies that the constant K_η given by (4.35) applied to $V_{0,\eta}$ may be written $K_\eta = K'\big(1+\frac{1}{\eta}\big)$, for some constant K' depending only on $\|\partial_{xx}V_0\|_\infty$. Therefore if we apply the Theorem 2.3 for the potential $V_{0,\eta}$, we get the bound

$$\mathcal{L}_{\varepsilon,\eta}^O(t) \le e^{2\|\partial_{xx}V_0\|_{L^\infty}t}\Big[\mathcal{L}_{\varepsilon,\eta}^O(0) + K'\varepsilon\Big(1+\frac{1}{\eta}\Big)\big(1+\mathcal{E}_{\varepsilon,0}+\mathcal{Q}_{\varepsilon,0}\big)\Big].$$

But we also have

$$\begin{aligned}
\mathcal{L}^O_{\varepsilon,\eta}(t) &= H_Q(\tilde f_\varepsilon) + \frac{1}{2}\left\|\Big(\varepsilon\,\partial_x V_\varepsilon - \partial_x V_0 \cos\frac{t}{\varepsilon}\Big) + \cos\frac{t}{\varepsilon}\Big(\partial_x V_0 - \partial_x V_{0,\eta}\Big)\right\|_2^2,\\
&\le H_Q(\tilde f_\varepsilon) + \left\|\varepsilon\,\partial_x V_\varepsilon - \partial_x V_0\cos\frac{t}{\varepsilon}\right\|_2^2 + \left\|\partial_x V_0 - \partial_x V_{0,\eta}\right\|_2^2,\\
&\le 2\,\mathcal{L}^O_\varepsilon(t) + C\,\eta^2,
\end{aligned}$$

thanks to the bound satisfied by $V_{0,\eta}$. Similarly, we can also prove that $\mathcal{L}^O_\varepsilon(t) \le 2\,\mathcal{L}^O_{\varepsilon,\eta}(t) + C\,\eta^2$. All in all, we get that

$$\begin{aligned}
\mathcal{L}^O_\varepsilon(t) &\le 2\,\mathcal{L}^O_{\varepsilon,\eta}(t) + C\,\eta^2\\
&\le 2\,e^{2\|\partial_{xx}V_0\|_{L^\infty} t}\Big[\mathcal{L}^O_{\varepsilon,\eta}(0) + K'\varepsilon\Big(1+\frac{1}{\eta}\Big)\big(1+\mathcal{E}_{\varepsilon,0}+\mathcal{Q}_{\varepsilon,0}\big)\Big] + C\,\eta^2\\
&\le 4\,e^{2\|\partial_{xx}V_0\|_{L^\infty} t}\Big[\mathcal{L}^O_\varepsilon(0) + K'\varepsilon\Big(1+\frac{1}{\eta}\Big)\big(1+\mathcal{E}_{\varepsilon,0}+\mathcal{Q}_{\varepsilon,0}\big) + C\,\eta^2\Big],
\end{aligned}$$

where we recall that the value of C may change form line to line. The claimed result follows from the choice $\eta = \varepsilon^{\frac{1}{3}}\big(1+\mathcal{E}_{\varepsilon,0}+\mathcal{Q}_{\varepsilon,0}\big)^{\frac{1}{3}}$.

4.5 Some analogies.

We conclude this section with a short digression about the analogies between the quasineutral limit and two classical singular limits in hydrodynamics.

The first one is the so-called hydrostatic approximation of the Euler equation. This limit turns out to be false in general due to the existence of instabilities for the unscaled system (see for instance [15]). As for the quasineutral limit, it is important to consider data satisfying a stability condition, namely the Rayleigh condition (a kind of monotonicity condition; we refer to [15] for details). There are similarities between our approach and the proof of derivation that Brenier gave in [17]. His proof relies also on some modulated energy method. As for the quasineutral limit, the first proof of this result was due to Grenier [30]; the techniques he used are related to those suggested in [29] (see also Masmoudi-Wong [46]).

There are also analogies with the derivation of the Prandtl equation in the inviscid limit of the Navier-Stokes equations and with its ill-posedness properties; we refer to [48, 51, 31, 24, 26, 32, 45, 25]. It would be very interesting to further investigate these.

5 Locally symmetric solutions to (1.3) are homogeneous.

In this section, we prove the following

Proposition 5.1. *Let f be a weak solution to the quasineutral equation* (1.3) *satisfying the following hypothesess. The electric field $E = -\partial_x V$ belongs to $L^\infty_t L^1_x$ and there exists a C^1 function $\bar v(t,x)$ such that*

i) for all t, x, $v \mapsto f(t,x,v)$ is increasing for $v < \bar v(t,x)$ and decreasing for $v > \bar v(t,x)$, and so has a maximum at $v = \bar v(t,x)$,

ii) for all t, x, v, $f(t,x,2\bar v(t,x) - v) = f(t,x,v)$.

Then there exist a constant (in time and position) $\bar v$ and a profile $\varphi : \mathbb{R}^- \to \mathbb{R}^+$, nondecreasing and satisfying $\int_{\mathbb{R}^+} \varphi(u)\frac{du}{\sqrt{u}} = 1$, such that for all $t \ge 0, x \in \mathbb{T}, v \in \mathbb{R}$,

$$f(t,x,v) = \varphi\left(-\frac{|v-\bar v|^2}{2}\right). \tag{5.1}$$

Similar "rigidity" properties (in a different context) were also studied by Ben Abdallah and Dolbeault [9, Section 2.3].

Proof of Proposition 5.1. During this proof we shall use the following notation : when f (or one of its derivatives) stands without reference to the variables, this means $f = f(t,x,v)$. Otherwise the variables are explicitly written. By instance, we will write the point *ii*) as $f(t,x,2\bar{v}(t,x)-v) = f$. For the function $\bar{v}$ there is no possible ambiguity since the variables will always be (t,x).

Step 1. Some remarkable identities.
We start form the equality $f(t,x,2\bar{v}-v) = f$, and differentiate it in v, t and x in order to get the following identities:

$$\begin{cases} \partial_v f(t,x,2\bar{v}-v) = -\partial_v f, \\ \partial_t f(t,x,2\bar{v}-v) = \partial_t f - 2\,\partial_v f(t,x,2\bar{v}-v)\,\partial_t \bar{v} = \partial_t f + 2\,\partial_v f\,\partial_t \bar{v}, \\ \partial_x f(t,x,2\bar{v}-v) = \partial_x f - 2\,\partial_v f(t,x,2\bar{v}-v)\,\partial_x \bar{v} = \partial_x f + 2\,\partial_v f\,\partial_x \bar{v}. \end{cases}$$

Using this in the equation (1.3) written at the point $(t,x,2\bar{v}-v)$, that is

$$\partial_t f(t,x,2\bar{v}-v) + (2\bar{v}-v)\,\partial_x f(t,x,2\bar{v}-v) + E\,\partial_v f(t,x,2\bar{v}-v) = 0,$$

we get

$$\partial_t f + (2\bar{v}-v)\partial_x f + [2\,\partial_t \bar{v} + 2(2\bar{v}-v)\partial_x \bar{v} - E]\,\partial_v f = 0\,.$$

Subtracting this equation to the original Vlasov equation (1.3) (and dividing by 2) we get

$$(v-\bar{v})\partial_x f + [E - \partial_t \bar{v} - (2\bar{v}-v)\partial_x \bar{v}]\,\partial_v f = 0\,.$$

Since the time does not appear in the equation, we may work for a fixed t.

Step 2. An ODE system at frozen t.
The previous equation means that at the frozen time t, the distribution $f(t,\cdot,\cdot)$ is constant along the trajectories of the system of ODEs

$$\begin{cases} \frac{d}{ds} X = \Xi - \bar{v}(t,X), \\ \frac{d}{ds} \Xi = E(t,X) - \partial_t \bar{v}(t,X) - (2\bar{v}(t,X) - \Xi)\partial_x \bar{v}(t,X). \end{cases}$$

Using the new variables $X, W := \Xi - \bar{v}(t,X)$, we get the system

$$\begin{cases} \frac{d}{ds} X = W, \\ \frac{d}{ds} W = E(t,X) - \partial_t \bar{v}(t,X) - \bar{v}(t,X)\partial_x \bar{v}(t,X) =: \tilde{E}(t,X). \end{cases}$$

We remark that $\bar{v}$ appears with the directional derivative $D_t = \partial_t + \bar{v}\partial_x$.

We now prove that $\tilde{E}$ is the gradient (with respect to x) of some potential $\tilde{V}$. Since we work in $1D$, it is sufficient to prove that the average of $\tilde{E}$ on $\mathbb{T}$ is equal to 0. Since $E = -\partial_x V$, we already know that $\int_{\mathbb{T}} E\,dx = 0$. We also have

$$\int_{\mathbb{T}} \bar{v}(t,x)\partial_x \bar{v}(t,x)\,dx = \int_{\mathbb{T}} \partial_x \left(\frac{\bar{v}(t,x)^2}{2} \right)\,dx = 0\,.$$

For the $\partial_t \bar{v}$ term, remark that from the symmetry assumption *ii*) and $\rho = 1$, we have

$$\int_{\mathbb{T}} \bar{v}(t,x)\,dx = \int_{\mathbb{T}} vf\,dxdv = P,$$

where P is the total momentum which is preserved by the equation (1.3). Therefore, the integral $\int_{\mathbb{T}} \partial_t \bar{v}(t,x)\, dx = \frac{dP}{dt}$ also vanishes.

We remark that according to [36] and since $\tilde{E} = -\partial_x \tilde{V} \in L^1$, the measure preserving flow associated to that system of ODEs is uniquely defined, and allows to construct the solutions of the associated transport equation.

Consequently, in the (X, W) coordinates, the trajectories are the curves $\tilde{V}(X) + \frac{W^2}{2} = Cst$. If $\bar{v}$ is C^1 and $E \in L^1_x$, then the potential $\tilde{V}$ is continuous. Since it is defined up to a constant, we assume that its minimum value is 0 and denote by x_0 a point where $\tilde{V}(x_0) = 0$. Using the assumptions $i)$ and $ii)$, we can define a function $g : \mathbb{R}^+ \to \mathbb{R}^+$, which is nonincreasing on $\mathbb{R}^+$ and such that for all $w \in \mathbb{R}$,

$$g\left(\frac{w^2}{2}\right) = f(t, x_0, \bar{v}(t, x_0) + w).$$

Let $(x, w) \in \mathbb{T} \times \mathbb{R}$. Since the integral curve of (x, w) crosses the line $\{x = x_0\}$, precisely at the point (x_0, w_0), where $\frac{w_0^2}{2} = \tilde{V}(x) + \frac{w^2}{2}$, we get that

$$f(t, x, \bar{v} + w) = g\left(\frac{w^2}{2} + \tilde{V}(x)\right). \tag{5.2}$$

Step 3. Using the $\rho = 1$ constraint.

Now, by (1.3), since for all x, we have $\rho(t, x) = 1$, we should have

$$\int f(t, x, v)\, dv = 2\int_0^{+\infty} g\left(\frac{w^2}{2} + \tilde{V}(x)\right) dw = 1.$$

This implies that $\tilde{V}(x) = 0$ for all x. Indeed, assume by contradiction that there exist x such that $\tilde{V}(x) > \tilde{V}(x_0) = 0$. Then, the following holds

$$\int_0^{+\infty} \left[g\left(\frac{w^2}{2}\right) - g\left(\frac{w^2}{2} + \tilde{V}(x)\right)\right] dw = 0.$$

Since $g(\cdot)$ is nonincreasing, we deduce that for all $w \in \mathbb{R}^+$,

$$g\left(\frac{w^2}{2}\right) - g\left(\frac{w^2}{2} + \tilde{V}(x)\right) = 0$$

But since g is integrable with the weight $\frac{1}{\sqrt{u}}$, g cannot be constant and there is $z \in \mathbb{R}^+$ such that

$$g\left(\frac{z^2}{2} + \tilde{V}(x)\right) < g\left(\frac{z^2}{2}\right).$$

This is a contradiction; we deduce that $\tilde{V} = 0$. This implies that $\tilde{E} = 0$ and thus we get the remarkable identity for the electric field

$$E(t, x) = \partial_t \bar{v}(t, x) + \bar{v}(t, x)\partial_x \bar{v}(t, x). \tag{5.3}$$

The ODE for (X, W) is now trivial, and we get a distribution $f(t, \cdot, \cdot)$ depending only on t and $|w|$, i.e.

$$f(t, x, \bar{v}(t, x) + w) = g\left(t, \frac{w^2}{2}\right)$$

or equivalently

$$f(t,x,v) = g\left(t, \frac{|v-\bar{v}(t,x)|^2}{2}\right). \tag{5.4}$$

Step 4. Consequences in the original Vlasov equation (1.3).
Inserting (5.4) in (1.3), it comes

$$\partial_t g\left(t, \frac{|v-\bar{v}(t,x)|^2}{2}\right) + \left[E - \partial_t \bar{v} - v\,\partial_x \bar{v}\right](v-\bar{v}(t,x))\partial_u g\left(t, \frac{|v-\bar{v}(t,x)|^2}{2}\right) = 0.$$

where $\partial_u g$ denotes the derivative with respect to the second variable of g. Using (5.3), we deduce

$$\partial_t g\left(t, \frac{|v-\bar{v}(t,x)|^2}{2}\right) - |\overline{v}-v|^2 \partial_x \bar{v}\,\partial_u g\left(t, \frac{|v-\bar{v}(t,x)|^2}{2}\right) = 0,$$

from which we can finally write

$$\partial_t g - 2\,u\,\partial_x \bar{v}\,\partial_u g = 0. \tag{5.5}$$

We can now integrate this equation against the measure $\frac{du}{\sqrt{u}}$, which yields (after an integration by parts)

$$\frac{d}{dt}\int g\frac{du}{\sqrt{u}} + \partial_x \bar{v}\int g\frac{du}{\sqrt{u}} = 0.$$

Recall that by (5.4), since $\rho = 1$, we have $\int g\frac{du}{\sqrt{u}} = 1$. We therefore deduce that for all t, x,

$$\partial_x \bar{v} = 0.$$

The equation (5.5) also becomes $\partial_t g = 0$ so that we can rewrite f as

$$f(t,x,v) = g\left(\frac{|v-\bar{v}(t)|^2}{2}\right).$$

But in this case $\bar{v}(t) = \int f\,dxdv = P$ which is preserved by the equation and therefore is a constant. We finally get

$$f(t,x,v) = g\left(\frac{|v-\bar{v}|^2}{2}\right)$$

and set $\varphi(\cdot) = g(-\cdot)$, which concludes the proof of (5.1). $\square$

6 Construction of BGK waves: Proof of Theorem 2.4

In this section, we give a proof of Theorem 2.4. We will restrict ourselves for simplicity to the case where $f_0^- = 0$, but the general case may be handled exactly in the same way. In our model, the Hamiltonian (or energy) is given by

$$E(x,v) = \frac{v^2}{2} + V(x),$$

and a solution of (2.19) is constant on the trajectories of the associated Hamiltonian system, which are the connected components of $\{(x,v)|E(x,v) = h\}$, except in the case where the potential remains constant on a whole interval.

We consider potentials V reaching their maximum at $x = 0$ and $x = 1$, i.e.

$$V(0) = V(1) = 0, \qquad \sup_{x\in[0,1]} V(x) \leq 0. \tag{6.1}$$

There exists at least one solution satisfying the above condition : the homogeneous equilibrium

$$f(x,v) = \begin{cases} f_0^+(v) & \text{if } v \geq 0, \\ 0 & \text{else,} \end{cases} \tag{6.2}$$

together with the constant potential $V = 0$. Under the condition (6.1), if $\frac{v^2}{2} + V(x) \geq 0$ and $v > 0$, then the level line passing through (x,v) crosses the incoming boundary $\{0\} \times \mathbb{R}^+$ at $(0, \sqrt{v^2 + 2V(x)})$. Therefore, the value of f at that point is given by

$$f(x,v) = f_0^+\big(\sqrt{v^2 + 2V(x)}\big), \qquad \text{if} \quad v \geq \sqrt{-2V(x)}.$$

If $v < 0$ and $\frac{v^2}{2} + V(x) \geq 0$, the level line crosses the incoming boundary $\{1\} \times \mathbb{R}^-$, and this leads to

$$f(x,v) = 0, \qquad \text{if} \quad v \leq -\sqrt{-2V(x)}.$$

In between, for $|v| \leq \sqrt{-2V(x)}$, the particles are "trapped" in the sense that they do not have a sufficient energy to reach one of the boundary, and as a consequence, their density is not fixed by the boundary condition. In that region, we will assume that the density of f is constant on the level lines of E, even if they are not connected. But this is not a restriction since there is only one density profile for the trapped particles that leads to a solution of (2.19) when $V(x)$ is known, as we shall see later. In order to be consistent with the previous discussion, we will use the notation

$$f(x,v) = f_T\big(\sqrt{-v^2 - 2V(x)}\big), \qquad \text{if} \quad |v| < \sqrt{-2V(x)}.$$

The unknown function f_T is defined on the interval $[0, \sqrt{-2V_{min}}]$, and the subscript T stands for "trapped". With these notation, the neutrality condition $1 = \int f\, dv$ now reads

$$1 = 2\int_0^{\sqrt{-2V(x)}} f_T\big(\sqrt{-v^2 - 2V(x)}\big)\, dv + \int_{\sqrt{-2V(x)}}^{+\infty} f_0^+\big(\sqrt{v^2 + 2V(x)}\big)\, dv.$$

After a change of variable, it may be rewritten

$$1 = 2\int_0^{\sqrt{-2V(x)}} f_T(u) \frac{u\, du}{\sqrt{-u^2 - 2V(x)}} + \int_0^{+\infty} f_0^+(u) \frac{u\, du}{\sqrt{u^2 - 2V(x)}}.$$

In order to get a solution, we only need to ensure that this condition is satisfied for any $x \in [0,1]$. This is precisely stated in the following lemma.

Lemma 6.1. *Assume that $f_0^+ \in L^1$ satifies $\int_0^{+\infty} f_0^+(v)\, dv = 1$, and that there exists a non-negative mesurable function $f_T : (0, +\infty) \to \mathbb{R}^+$ such that*

$$\forall\, r > 0, \quad 2\int_0^r f_T(u) \frac{u\, du}{\sqrt{r^2 - u^2}} = 1 - \int_0^{+\infty} f_0^+(u) \frac{u\, du}{\sqrt{r^2 + u^2}}. \tag{6.3}$$

Then, for any continuous potential $V : [0,1] \to \mathbb{R}^-$ satisfying $V(0) = V(1) = 0$, the function $f \in L^1$ defined by

$$f(x,v) = \begin{cases} f_0^+\big(\sqrt{v^2 + 2V(x)}\big) & \text{if} \quad v \geq \sqrt{-2V(x)}, \\ 0 & \text{if} \quad v \leq -\sqrt{-2V(x)}, \\ f_T\big(-\sqrt{-v^2 - 2V(x)}\big) & \text{if} \quad |v| < \sqrt{-2V(x)}, \end{cases} \tag{6.4}$$

is a solution of (2.19).

Proof of Lemma 6.1. The proof is straightforward. The condition (6.3) ensures that $\rho(x) = 1$, and in particular it implies that (6.4) defines a function $f \in L^1$. The fact that the function f solves (2.19) in the sense of distributions follows since f is a function of the energy $\frac{v^2}{2} + V(x)$. It can be checked using some smooth test function φ, and the change of variables $(x, v) \mapsto \big(x, v^2 + 2V(x)\big)$ in several regions. $\square$

So in order to construct solutions to (2.19), it suffices to find a function f_T satisfying (6.3). This is done in the following proposition.

Proposition 6.1. *Assume that $f_0^+ \in L^1$ with $\int f_0^+ = 1$. Then the function f_T, defined on $(0, +\infty)$ as follows*

$$f_T(u) := \frac{1}{\pi} \int_0^\infty f_0^+(v) \frac{u\, v\, dv}{(u^2 + v^2)^{\frac{3}{2}}}, \tag{6.5}$$

satisfies (6.3) for all $r > 0$. And for any $\bar{r} > 0$, it is the unique function that satisfies (6.3) for all $r \in [0, \bar{r}]$. Moreover, if in addition f_0^+ is continuous at 0, then $\lim_{u \to 0} f_T(u) = f_0^+(0)$.

The theorem 2.4 is then a consequence of Lemma 6.1 and Proposition 6.1.

Proof of Proposition 6.1.
Step 1. f_T is a solution of (6.3). First remark that the function f_T defined by (6.5) may be rewritten

$$f_T(u) = \frac{1}{2\pi} \int_0^\infty f_0^+(uv) \frac{dv}{(1 + v^2)^{\frac{3}{2}}}.$$

This allows to prove that $\lim_{u \to 0} f_T(u) = f_0^+(0)$ when f_0^+ is continuous at 0. Next, denote by g the function defined on $\mathbb{R}^+$ by the r.h.s. of (6.3):

$$g(r) := 1 - \int_0^{+\infty} f_0^+(u) \frac{u\, du}{\sqrt{r^2 + u^2}}. \tag{6.6}$$

Its derivative is given for $r > 0$ by

$$g'(r) = r \int_0^\infty \frac{f_0^+(u)\, u\, du}{(r^2 + u^2)^{\frac{3}{2}}} = \int_0^\infty \frac{f_0^+(ru)\, u\, du}{(1 + u^2)^{\frac{3}{2}}}.$$

Thus g' is positive and thus in L^1, and with the second expression, we see that $\lim_{r \to 0} g'(r) = f_0^+(0)$ if f_0^+ is continuous at 0. The condition (6.3) maybe rewritten

$$\forall\, r \in [0, \bar{r}], \qquad \int_0^r f_T(u) \frac{u\, du}{\sqrt{r^2 - u^2}} = \frac{1}{2}\, g(r).$$

Next using that for all $a > b \geq 0$,

$$\int_a^b \frac{u\, du}{\sqrt{(b^2 - u^2)(u^2 - a^2)}} = \frac{\pi}{2}, \tag{6.7}$$

we obtain that for all $r > 0$

$$\int_0^r \left(\frac{1}{\pi} \int_0^u \frac{g'(s)\, ds}{\sqrt{u^2 - s^2}} \right) \frac{u\, du}{\sqrt{r^2 - u^2}} = \frac{1}{2} \int_0^r g'(s)\, ds = \frac{1}{2}\, g(r).$$

This means that

$$f_T(u) := \frac{1}{\pi}\int_0^u \frac{g'(s)\,ds}{\sqrt{u^2-s^2}} = \frac{1}{\pi}\int_0^u\int_0^\infty \frac{f_0^+(v)\,s\,v\,ds\,dv}{\sqrt{(u^2-s^2)(s^2+v^2)^3}}$$
$$= \frac{1}{2\pi}\int_0^1\int_0^\infty \frac{f_0^+(uv)\,v\,ds\,dv}{\sqrt{(1-s)(s+v^2)^3}},$$

is a solution to (6.3). The conclusion follows from the equality

$$\int_0^1 \frac{ds}{\sqrt{(1-s)(s+x)^3}} = \left[-\frac{2\sqrt{1-s}}{(1+x)\sqrt{x+s}}\right]_0^1 = \frac{2}{(1+x)^{\frac{3}{2}}}.$$

Step 2. Uniqueness. Assume that $g_T : [0,\bar{r}] \to \mathbb{R}$ is such that

$$\forall r \in [0,\bar{r}], \qquad \int_0^r g_T(u)\frac{u\,du}{\sqrt{r^2-u^2}} = 0.$$

Let $r_1 \in [0,r]$. Then, multiplying the previous equation by $\frac{r}{\sqrt{r_1^2-r^2}}$, and integrating on the interval $[0,r_1]$, and using (6.7), we get

$$\frac{\pi}{2}\int_0^{r_1} g_T(u)u\,du = 0.$$

Since it holds for any $r_1 \in [0,\bar{r}]$, it implies that $g_T = 0$ on that interval.

□

7 The case of the Vlasov-Poisson equation for ions

In the last part of this paper, we focus on the quasineutral limit of the Vlasov-Poisson equation for ions:

$$\begin{cases} \partial_t f_\varepsilon + v\partial_x f_\varepsilon - \partial_x V_\varepsilon \partial_v f_\varepsilon = 0, \\ \alpha V_\varepsilon - \varepsilon^2\,\partial_x^2 V_\varepsilon = \rho_\varepsilon - 1, \end{cases} \tag{7.1}$$

where $\alpha > 0$. We add an initial condition $f_{0,\varepsilon} \in L^1$ such that $f_{0,\varepsilon} \geq 0$, $\int f_{0,\varepsilon} dvdx = 1$. As already said in the introduction, this allows to describe the dynamics of ions in a plasma, in a background of "adiabatic" electrons, i.e. electrons which instantaneously reach a thermodynamic equilibrium.

Remark 7.1. *In* (7.1)*, there is a parameter $\alpha > 0$, which comes from the fact that this model is only a linearization of the "physical" equations, in which the density of electrons follows a Maxwell-Boltzmann law and the Poisson equation thus reads:*

$$-\varepsilon^2\,\partial_x^2 V_\varepsilon = \rho_\varepsilon - e^{-\alpha V_\varepsilon}.$$

The linearization consists then in writing $e^{-\alpha V_\varepsilon} \approx 1 - \alpha V_\varepsilon$, which yields (7.1)*.*

The scaled physical energy of this system reads:

$$\mathcal{E}_\varepsilon(t) = \frac{1}{2}\int f_\varepsilon|v|^2 dvdx + \frac{\alpha}{2}\int V_\varepsilon^2 dx + \frac{\varepsilon^2}{2}\int |\partial_x V_\varepsilon|^2 dx. \tag{7.2}$$

Assume now to simplify that $\alpha = 1$. We can proceed as in the introduction and formally obtain in the limit $\varepsilon \to 0$ the Vlasov-Dirac-Benney equation

$$\begin{cases} \partial_t f + v\partial_x f - \partial_x V \partial_v f = 0, \\ V = \rho - 1. \end{cases} \tag{7.3}$$

We observe that the energy associated to this system reads :

$$\mathcal{E}(t) = \frac{1}{2}\int f|v|^2 dv dx + \frac{1}{2}\int \rho^2 dx. \tag{7.4}$$

Remark 7.2. *Note that this can be seen a kinetic version of the shallow water system (or isentropic gas dynamics with $\gamma = 2$). Indeed, for monokinetic profiles, that is*

$$f(t,x,v) = \rho(t,x)\delta_{v=u(t,x)},$$

we get the one-dimensional shallow water system:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t u + u\partial_x u + \partial_x \rho = 0. \end{cases} \tag{7.5}$$

As a matter of fact, the derivation of (7.5) *from* (7.1) *for monokinetic data was performed in [35].*

We now explain how to adapt the results which have been proved in this paper.

7.1 The Penrose criterion.

We shall start by explaining what is the right Penrose criterion in the context of the Vlasov equation

$$\begin{cases} \partial_t f + v\partial_x f - \partial_x V \partial_v f = 0, \\ \alpha V - \partial_x^2 V = \rho - 1. \end{cases} \tag{7.6}$$

We define the "α-Penrose instability criterion" as follows.

Definition 7.1. *We say that an homogeneous profile $\mu(v)$, such that $\int \mu \, dv = 1$, satisfies the α-Penrose instability criterion if there exists a local minimum point $\bar{v}$ of μ such that the following inequality holds*

$$\int_{\mathbb{R}} \frac{\mu(v) - \mu(\bar{v})}{(v-\bar{v})^2} \, dv > \alpha. \tag{7.7}$$

If the local minimum is flat, i.e. is reached on an interval $[\bar{v}_1, \bar{v}_2]$, then (7.7) *has to be satisfied for all $\bar{v} \in [\bar{v}_1, \bar{v}_2]$.*

Remark 7.3. *Note that if $\alpha = 0$, we recover the same instability conditions of the introduction.*

Exactly as for the case $\alpha = 0$, we may obtain the exact analogue of Proposition 3.2, which is a key point in Theorem 7.1 below.

7.2 Unstable Case.

The instability result we are able to prove is the same as for $\alpha = 0$.

Theorem 7.1. *Let $\mu(v)$ be a smooth positive profile satisfying the Penrose instability criterion of Definition 7.1 and the δ-condition of Definition* (2.1)*. For any $N > 0$ and $s > 0$, there exists a sequence of non-negative initial data $(f_{0,\varepsilon})$ such that*

$$\|f_{\varepsilon,0} - \mu\|_{W^{s,1}_{x,v}} \leq \varepsilon^N,$$

and denoting by (f_ε) the sequence of solutions to (7.1) *with initial data $(f_{0,\varepsilon})$, the following holds:*

i) L^1 **instability for the macroscopic observables:** *the density $\rho_\varepsilon := \int f_\varepsilon \, dv$, and the electric field $E_\varepsilon = -\partial_x V_\varepsilon$. For all $\alpha \in [0, 1)$, we have*

$$\liminf_{\varepsilon \to 0} \sup_{t \in [0,\varepsilon^\alpha]} \|\rho_\varepsilon(t) - 1\|_{L^1_x} > 0, \qquad \liminf_{\varepsilon \to 0} \sup_{t \in [0,\varepsilon^\alpha]} \varepsilon \|E_\varepsilon\|_{L^1} > 0. \tag{7.8}$$

ii) **Full instability for the distribution function:** *for any $r \in \mathbb{Z}$, we have*

$$\liminf_{\varepsilon \to 0} \sup_{t \in [0,\varepsilon^\alpha]} \|f_\varepsilon(t) - \mu\|_{W^{r,1}_{x,v}} > 0 \tag{7.9}$$

The same proof as that of Theorem 2.1 holds, *mutatis mutandis*: we only have to switch the Poisson equations.

7.3 Stable Case.

The following holds for any value of α, and therefore we consider here for simplicity that $\alpha = 1$. We restrict ourselves only on the well-prepared case. Our stability theorem for (7.1) goes as follows:

Theorem 7.2. *Let μ be a S-stable stationary solution to* (7.3) *of the form given in* (2.8)*. Assume that there exists $\eta > 0$, such that μ satisfies*

$$\int \mu(v)(1 + v^{2+\eta}) \, dv < +\infty. \tag{7.10}$$

For all $\varepsilon > 0$, let $(f_\varepsilon, V_\varepsilon)$ be the global weak solution in the sense of Arsenev to (7.1)*, with initial datum $f_{0,\varepsilon}$ and define the "modulated energy"*

$$\mathcal{L}_\varepsilon[f_\varepsilon] := H_Q(f_\varepsilon) + \frac{\varepsilon^2}{2} \int (\partial_x V_\varepsilon)^2 dx + \frac{1}{2} \int V_\varepsilon^2 dx. \tag{7.11}$$

Then, $\mathcal{L}_\varepsilon$ is a Lyapunov functional in the sense that

$$\forall t \in \mathbb{R}^+, \qquad \mathcal{L}_\varepsilon[f_\varepsilon(t)] = \mathcal{L}_\varepsilon[f_{0,\varepsilon}].$$

We thus see that the only thing to do is to adapt the definition of the modulated energy (7.11) according to the energy (7.2) of (7.1). Then, the proof is exactly the same as that of Theorem 2.2, and therefore we omit it.

The adaptation the stability result for ill-prepared initial data of Theorem 2.3 requires more than a simple rephrasing (even the formal derivation of the plasma oscillations is not so clear in this case), and this does not fall within the scope of this paper.

7.4 Construction of the associated BGK waves.

We consider now the boundary value problem

$$\begin{cases} v\,\partial_x f - \dfrac{1}{\alpha}\partial_x \rho\,\partial_v f = 0, \\ \rho = \displaystyle\int f(x,v)\,dv, \end{cases} \tag{7.12}$$

on the space $\Omega = [0,1] \times \mathbb{R}$. The incoming boundary conditions are given by (2.20). This corresponds to the stationary equations associated to the Vlasov-Dirac-Benney equation (7.1), with boundary conditions. An adaptation of the proof of Theorem 2.4 leads to the following theorem.

Theorem 7.3. *Assume that $f_0^\pm : \mathbb{R}^+ \to \mathbb{R}^+$ is are nonnegative and measurable functions such that $\int_0^{+\infty} \big(f_0^+(v) + f_0^-(-v)\big)\,dv = 1$. Define f_T on $(0,+\infty)$ by*

$$f_T(u) := -\frac{\alpha\,u}{2\pi} + \frac{1}{\pi}\int_0^\infty \big(f_0^+(v) + f_0^-(-v)\big)\frac{u\,v\,dv}{(u^2+v^2)^{\frac{3}{2}}}, \tag{7.13}$$

and denote $\bar{u} := \inf\{u > 0,\ s.t.\ f_T(u) < 0\}$, which is in any case bounded by $\sqrt{\frac{2}{\alpha}}$. Then, for any continuous potential with values in $\big[-\frac{\bar{u}^2}{2}, 0\big]$ satisfying $V(0) = V(1) = 0$, the function f defined by (2.22) together with V gives a solution of (7.12) in the sense of distributions. Moreover, any solution with V nonpositive and vanishing at the boundary is of the above form.

The bound on $\bar{u}$ comes from a straightforward a priori bound of the right hand side of (7.13): use the elementary inequality $(u^2+v^2)^{-\frac{3}{2}} \le u^{-2}v^{-1}$ and $\int_0^{+\infty}\big(f_0^+(v) + f_0^-(-v)\big)\,dv = 1$.

When $f_0^+(\cdot) + f_0^-(-\cdot)$ is continuous at 0, we have also that $\lim_{u\to 0} f_T(u) = f_0^+(0) + f_0^-(0)$, so that it is clear that $\bar{u}$ is strictly positive if $f_0^+(0) + f_0^-(0)$ is. If $f_T(0) = 0$, then $\bar{u} > 0$ when $f_T'(u) > 0$, that is

$$\int_0^\infty \big(f_0^+(v) + f_0^-(-v)\big)\frac{dv}{v^2} > \frac{\alpha}{2}.$$

The proof of 2.4 can be adapted without difficulty to this case, with a potential given by $V = \frac{\rho-1}{\alpha}$. For instance, the definition of g in (6.6) should be replaced by

$$g(r) := 1 - \alpha\frac{r^2}{2} + \int_0^{+\infty} f_0^+(u)\frac{u\,du}{\sqrt{r^2+u^2}},$$

and the additionnal term $-\alpha\frac{r^2}{2}$ leads after some straightforward computations to the additional term $-\frac{\alpha\,u}{2\pi}$ in (7.13).

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