On the three-dimensional finite Larmor radius approximation: the case of electrons in a fixed background of ions

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Abstract

This paper is concerned with the analysis of a mathematical model arising in plasma physics, more specifically in fusion research. It directly follows [18], where the three-dimensional analysis of a Vlasov-Poisson equation with finite Larmor radius scaling was led, corresponding to the case of ions with massless electrons whose density follows a linearized Maxwell-Boltzmann law. We now consider the case of electrons in a background of fixed ions, which was only sketched in [18]. Unfortunately, there is evidence that the formal limit is false in general. Nevertheless, we formally derive from the Vlasov-Poisson equation a fluid system for particular monokinetic data. We prove the local in time existence of analytic solutions and rigorously study the limit (when the inverse of the intensity of the magnetic field and the Debye length vanish) to a new anisotropic fluid system. This is achieved thanks to Cauchy-Kovalevskaya type techniques, as introduced by Caflisch [7] and Grenier [14]. We finally show that this approach fails in Sobolev regularity, due to multi-fluid instabilities.

Keywords: Gyrokinetic limit - Finite Larmor Radius Approximation - Anisotropic quasineutral limit - Anisotropic hydrodynamic systems - Analytic regularity - Cauchy-Kovalevskaya theorem - Ill-posedness in Sobolev spaces.

1 Introduction

1.1 Presentation of the problem

The main goal of this paper is to derive some fluid model in order to understand the behaviour of a quasineutral gas of electrons in a neutralizing background of fixed ions and submitted to a strong external magnetic field. For simplicity, we consider that the magnetic field has fixed direction and intensity. The density of the electrons is governed by the classical Vlasov-Poisson equation. We first introduce some notations:

Notations. *Let* $(e_1, e_2, e_\parallel)$ *be a fixed orthonormal basis of* $\mathbb{R}^3$.

- *The subscript* $\perp$ *stands for the orthogonal projection on the plane* (e_1, e_2)*, while the subscript* $\parallel$ *stands for the projection on* $e_\parallel$.
- *For any vector* $X = (X_1, X_2, X_\parallel)$*, we define* $X^\perp$ *as the vector* $(X_2, -X_1, 0) = X \wedge e_\parallel$.
- *We define the differential operators* $\Delta_{x_\parallel} = \partial^2_{x_\parallel}$ *and* $\Delta_{x_\perp} = \partial^2_{x_1} + \partial^2_{x_2}$.

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Then the magnetic field we consider can be taken as:

$$B = \overline{B} e_{\|},$$

where $\overline{B} > 0$ is a constant. In order to describe the *turbulent* behaviour of the plasma (we refer to the appendix for physical explanations), we study the following scaled Vlasov-Poisson system (for $t > 0, x \in \mathbb{T}^3 := \mathbb{R}^3/\mathbb{Z}^3, v \in \mathbb{R}^3$ and ϵ is a small positive constant):

$$\begin{cases} \partial_t f_\epsilon + \frac{v_\perp}{\epsilon}.\nabla_x f_\epsilon + v_\|.\nabla_x f_\epsilon + (E_\epsilon + \frac{v \wedge e_\|}{\epsilon}).\nabla_v f_\epsilon = 0 \\ E_\epsilon = (-\nabla_{x_\perp} V_\epsilon, -\epsilon \nabla_{x_\|} V_\epsilon) \\ -\epsilon^2 \Delta_{x_\|} V_\epsilon - \Delta_{x_\perp} V_\epsilon = \int f_\epsilon dv - \int f_\epsilon dv dx \\ f_{\epsilon, t=0} = f_{\epsilon,0} \geq 0, \quad \int f_{\epsilon,0} dv dx = 1. \end{cases} \tag{1.1}$$

The non-negative quantity $f_\epsilon(t, x, v)$ is interpreted as the distribution function of the electrons: this means that $f_\epsilon(t, x, v) dx dv$ is the probability of finding particles at time t with position x and velocity v; $V_\epsilon(t, x)$ and $E_\epsilon(t, x)$ are respectively the electric potential and force. Finally, $\frac{v \wedge e_\|}{\epsilon}$ corresponds to the Lorentz force and is due to the magnetic field B.

This corresponds to the so-called finite Larmor radius scaling for the Vlasov-Poisson equation, which was introduced by Frénod and Sonnendrücker in the mathematical literature [10]. The $2D$ version of the system (obtained when one restricts to the perpendicular dynamics) and the limit $\epsilon \to 0$ were studied in [10] and more recently in [3, 11, 9]. We also refer to the recent work [20] of Hauray and Nouri, dealing with the wellposedness theory with a diffusive version of a related $2D$ system.

A version of the full $3D$ system describing ions with massless electrons was studied by the author in [18]. In this former work, we considered that the density of electrons follows a linearized Maxwell-Boltzmann law. This means that we studied the following Poisson equation for the electric potential:

$$V_\epsilon - \epsilon^2 \Delta_{x_\|} V_\epsilon - \Delta_{x_\perp} V_\epsilon = \int f_\epsilon dv - \int f_\epsilon dv dx. \tag{1.2}$$

In this case it was shown after some filtering that the number density f_ϵ weakly converges as $\epsilon \to 0$ to some solution f to another kinetic system exhibiting the so-called $E \times B$ drift in the orthogonal plane, but with trivial dynamics in the parallel direction. This last feature seemed somehow disappointing.

We observed in [18] that in the case where the Poisson equation reads (which precisely corresponds to the case of (1.1)):

$$-\epsilon^2 \Delta_{x_\|} V_\epsilon - \Delta_{x_\perp} V_\epsilon = \int f_\epsilon dv - \int f_\epsilon dv dx, \tag{1.3}$$

we could expect to make a pressure appear in the limit process $\epsilon \to 0$, due to some incompressibility constraint. Indeed, passing formally to the limit $\epsilon \to 0$ (and assuming that f_ϵ converges to f and V_ϵ converges to V in some sense), we obtain:

$$-\Delta_{x_\perp} V = \int f dv - \int f dv dx,$$

and integrating this equation with respect to $x_\perp$, we finally get the incompressibility constraint:

$$\int f dv dx_\perp = \int f dv dx.$$

Unfortunately, we were not able to rigorously derive a kinetic limit or even a fluid limit from (1.1). This is not only due to technical mathematical difficulties. This is related to the existence of instabilities for the Vlasov-Poisson equation, such as the double-humped instabilities (see Guo and Strauss [16]) and their counterpart in the multi-fluid Euler equations, such as the two-stream instabilities (see Cordier, Grenier and Guo [8]). Such instabilities actually take over in the limit $\epsilon \to 0$ and the formal limit is false in general, unless $f_{\epsilon,0}$ does not depend on parallel variables, which corresponds to the $2D$ problem studied by Frénod and Sonnendrücker [10].

Actually, we can observe that if on the contrary the initial data $f_{\epsilon,0}$ depends only on parallel variables, we obtain the one-dimensional quasineutral system (the first equation is simply the one-dimensional Vlasov equation, note that there is no more magnetic field):

$$\left\{ \begin{array}{l} \partial_t f_\epsilon + v_\| \partial_{x_\|} f_\epsilon - \partial_{x_\|} V_\epsilon \partial_{v_\|} f_\epsilon = 0 \\ -\epsilon \partial^2_{x_\|} V_\epsilon = \int f_\epsilon dv - \int f_\epsilon dv dx_\| \\ f_{\epsilon,t=0} = f_{\epsilon,0} \geq 0, \quad \int f_{\epsilon,0} dv dx_\| = 1. \end{array} \right. \tag{1.4}$$

The formal limit is easily obtained, by taking $\epsilon = 0$:

$$\left\{ \begin{array}{l} \partial_t f + v_\| \partial_{x_\|} f - \partial_{x_\|} V \partial_{v_\|} f = 0 \\ \int f dv = \int f dv dx_\| \\ f_{t=0} = f_0 \geq 0, \quad \int f_0 dv dx_\| = 1. \end{array} \right. \tag{1.5}$$

In [15], an explicit example of Grenier shows that the formal limit is false in general, because of the double-humped instability:

Theorem 1.1 (Grenier, [15])**.** *We define an initial data f_0 by:*

$$\begin{aligned} f_0(x,v) &= 1 \quad \text{for} \quad -1 \leq v \leq -1/2 \text{ and } 1/2 \leq v \leq 1 \\ &= 0 \quad \text{elsewhere.} \end{aligned}$$

For any N and s in $\mathbb{N}$, and for any $\epsilon < 1$, there exist for $i = 1,2,3,4$, $v_i^\epsilon(x) \in H^s(\mathbb{T})$ with $\|v_1^\epsilon(x) + 1\|_{H^s} \leq \epsilon^N$, $\|v_2^\epsilon(x) + 1/2\|_{H^s} \leq \epsilon^N$, $\|v_3^\epsilon(x) - 1/2\|_{H^s} \leq \epsilon^N$, $\|v_4^\epsilon(x) - 1\|_{H^s} \leq \epsilon^N$, such that the solution $f_\epsilon(t,x,v)$ associated to the initial data defined by:

$$\begin{aligned} f_{\epsilon,0}(x,v) &= 1 \quad \text{for} \quad v_1^\epsilon(x) \leq v \leq v_2^\epsilon(x) \text{ and } v_3^\epsilon(x) \leq v \leq v_4^\epsilon(x) \\ &= 0 \quad \text{elsewhere,} \end{aligned}$$

does not converge to f_0 in the following sense:

$$\liminf_{\epsilon \to 0} \sup_{t \leq T} \int |f_\epsilon(t,x,v) - f_0(v)| v^2 dv dx > 0 \tag{1.6}$$

for any $T > 0$ and also for $T = \epsilon^\alpha$, with $\alpha < 1/2$.

In order to overcome the effects of these instabilities for the usual quasineutral limit, there are two possibilities:

- One consists in restricting to particular initial profiles chosen in order to be stable (this would imply in particular some monotony conditions on the data, such as the Penrose condition [25]).
- The other one consists in considering data with analytic regularity, in which case the instabilities (which turn out to be essentially of "Sobolev" nature) do not have any effect.

Here the situation is worst: by opposition to the usual quasineutral limit (see [6], [15]), restricting to stable profiles is not sufficient. This is due to the anisotropy of the problem and the dynamics in the perpendicular variables.

In this paper, we illustrate this phenomenon by studying the following fluid system, formally derived from the kinetic system (1.1) by considering some physically relevant monokinetic data (we refer to the appendix for the detailed formal derivation).

$$\begin{cases} \partial_t \rho_\epsilon + \nabla_\perp.(E_\epsilon^\perp \rho_\epsilon) + \partial_{\|}(v_{\|,\epsilon}\rho_\epsilon) = 0 \\ \partial_t v_{\|,\epsilon} + \nabla_\perp.(E_\epsilon^\perp v_{\|,\epsilon}) + v_{\|,\epsilon}\partial_{\|}(v_{\|,\epsilon}) = -\epsilon\partial_{\|}\phi_\epsilon(t,x) - \partial_{\|}V_\epsilon(t,x_{\|}) \\ E_\epsilon^\perp = -\nabla^\perp \phi_\epsilon \\ -\epsilon^2\partial_{\|}^2\phi_\epsilon - \Delta_\perp\phi_\epsilon = \rho_\epsilon - \int \rho_\epsilon dx_\perp \\ -\epsilon\partial_{\|}^2 V_\epsilon = \int \rho_\epsilon dx_\perp - 1, \end{cases} \tag{1.7}$$

where:

- $\rho_\epsilon(t, x_\perp, x_{\|}) : \mathbb{R}^+ \times \mathbb{T}^3 \to \mathbb{R}_*^+$ can be interpreted as a charge density,
- $v_{\|,\epsilon}(t, x_\perp, x_{\|}) : \mathbb{R}^+ \times \mathbb{T}^3 \to \mathbb{R}$ can be interpreted as a "parallel" current density.
- $\phi_\epsilon(t, x_{\|})$ and $V_\epsilon(t,x)$ are electric potentials.

Although we have considerered monokinetic data, (1.7) is intrinsically a "multi-fluid" system, because of the dependence on $x_\perp$. Hence, we still have to face the two-stream instabilities ([8]): because of these, the limit is false in Sobolev regularity and we thus decide to study the associated Cauchy problem for analytic data.

We then prove the limit to a new fluid system which is strictly speaking compressible but also somehow "incompressible in average". This rather unusual feature is due to the anisotropy of the model. The fluid system is the following (obtained formally by taking $\epsilon = 0$):

$$\begin{cases} \partial_t \rho + \nabla_\perp.(E^\perp \rho) + \partial_{\|}(v_{\|}\rho) = 0 \\ \partial_t v_{\|} + \nabla_\perp.(E^\perp v_{\|}) + v_{\|}\partial_{\|}(v_{\|}) = -\partial_{\|}p(t,x_{\|}) \\ E^\perp = \nabla^\perp \Delta_\perp^{-1}\left(\rho - \int \rho dx_\perp\right) \\ \int \rho dx_\perp = 1. \end{cases} \tag{1.8}$$

We observe that this system can be interpreted as an infinite system of Euler-type equations, coupled together through the "parameter" $x_\perp$ by the constraint:

$$\int \rho dx_\perp = 1.$$

It has some interesting features:

- This system is anisotropic in $x_\perp$ and $x_{\|}$ and it somehow combines two features of the incompressible Euler equations. The $2D$ part of the dynamics of the equation for ρ is nothing but the vorticity formulation of $2D$ incompressible Euler. Nevertheless, physically speaking, ρ should be interpreted here as a density rather than a vorticity. The dynamics in the parallel direction is similar to the dynamics of incompressible Euler written in velocity. We finally observe that the pressure p only depends on the parallel variable $x_{\|}$ and not on $x_\perp$.
- This does not strictly speaking describe an incompressible fluid, since $(E^\perp, v_{\|})$ is not divergence free. Somehow, the fluid is hence compressible. But the constraint $\int \rho dx_\perp = 1$ can be interpreted as a constraint of "incompressibility in average" which

allows one to recover the pressure law from the other unknowns. Indeed, we easily get, by integrating with respect to $x_\perp$ the equation satisfied by ρ:

$$\partial_{x_\parallel} \int \rho v_\parallel dx_\perp = 0. \tag{1.9}$$

So by plugging this constraint in the equation satisfied by $\rho v_\parallel$, that is:

$$\partial_t(\rho v_\parallel) + \nabla_\perp.(E^\perp \rho_\parallel v_\parallel) + \partial_\parallel(\rho v_\parallel^2) = -\partial_\parallel p(t, x_\parallel)\rho,$$

we get the (one-dimensional) elliptic equation allowing to recover $-\partial_{x_\parallel} p$:

$$-\partial_\parallel^2 p(t, x_\parallel) = \partial_\parallel^2 \int \rho v_\parallel^2 dx_\perp,$$

from which we get:

$$-\partial_\parallel p(t, x_\parallel) = \partial_\parallel \int \rho v_\parallel^2 dx_\perp. \tag{1.10}$$

- From the point of view of plasma physics, $E^\perp.\nabla_\perp$ corresponds to the so-called electric drift. By analogy with the so-called drift-kinetic equations [28], we can call this system a drift-fluid equation. To the best of our knowledge, this is the very first time such a model is exhibited in the literature.

From now on, when there is no risk of confusion, we will sometimes write v and v_ϵ instead of $v_\parallel$ and $v_{\parallel,\epsilon}$.

1.2 Organization of the paper

The outline of this paper is as follows. In Section 2, we will state the main results of this paper that are: the existence of analytic solutions to (1.7) locally in time but uniformly in ϵ (Theorem 2.1), the strong convergence to (1.8) with a complete description of the plasma oscillations (Theorem 2.2) and the existence and uniqueness of local analytic solutions to (1.8), in Proposition 2.1.

Section 3 is devoted to the proof of Theorem 2.1. First we recall some elementary features of the analytic spaces we consider (section 3.1), then we implement an approximation scheme for our Cauchy-Kovalesvkaya type existence theorem. The results are based on a decomposition of the electric field allowing for a good understanding of the so-called plasma waves (section 3.2).

In section 4, we prove Theorem 2.2, by using the uniform in ϵ estimates we have obtained in the previous theorem. The proof relies on another decomposition of the electric field, in order to exhibit the effects of the plasma waves as ϵ goes to 0.

Then, in section 5, we discuss the sharpness of our results:

- In sections 5.1 and 5.2, we discuss the analyticity assumption and explain why we can not lower down the regularity to Sobolev. In section 5.3, we explain why it is not possible to obtain global in time results. We obtain these results by considering some well-chosen initial data and using results of Brenier on multi-fluid Euler systems [5].
- Because of the two-stream instabilities, studying the limit with the relative entropy method is bound to fail. Nevertheless we found it interesting to try to apply the method and see at which point things get nasty: this is the object of section 5.4, where we study a kinetic toy model which retains the main unstable feature of system (1.7).

The two last sections are respectively a short conclusion and an appendix where we explain the scaling and the formal derivation of system (1.7).

2 Statement of the results

In order to prove both the existence of strong solutions to systems (1.7) and (1.8) and also prove the results of convergence, we follow the construction of Grenier [14], with some modifications adapted to our problem.

In [14], Grenier studies the quasineutral limit of the family of coupled Euler-Poisson systems:

$$\begin{cases} \partial_t \rho_\Theta^\epsilon + \operatorname{div}(\rho_\Theta^\epsilon v_\Theta^\epsilon) = 0 \\ \partial_t v_\Theta^\epsilon + v_\Theta^\epsilon . \nabla(v_\Theta^\epsilon) = E^\epsilon \\ \operatorname{rot} E^\epsilon = 0 \\ \epsilon \operatorname{div} E^\epsilon = \int_M \rho_\Theta^\epsilon \mu(d\Theta) - 1, \end{cases} \tag{2.1}$$

with (M, Θ, μ) a probability space.

Following the proof of the Cauchy-Kovalevskaya theorem given by Caflisch [7], Grenier proved the local existence of analytic functions (with respect to x) uniformly with respect to ϵ and then, after filtering the fast oscillations due to the force field, showed the strong convergence to the system:

$$\begin{cases} \partial_t \rho_\Theta + \operatorname{div}(\rho_\Theta v_\Theta) = 0 \\ \partial_t v_\Theta + v_\Theta^\epsilon . \nabla(v_\Theta) = E \\ \operatorname{rot} E = 0 \\ \int \rho_\Theta \mu(d\Theta) = 1. \end{cases} \tag{2.2}$$

We notice that the class of systems studied by Grenier is close to system (1.7), if we take $x = x_\parallel$, $\Theta = x_\perp$ and $(M, \mu) = (\mathbb{T}^2, dx_\perp)$, the main difference being that we have to deal with a dynamics in $\Theta = x_\perp$.

Hence, we introduce the same spaces of analytic functions as in [14], but this time depending also on $\Theta = x_\perp$.

Definition. *Let $\delta > 1$. We define B_δ the space of real functions ϕ on $\mathbb{T}^3$ such that*

$$|\phi|_\delta = \sum_{k \in \mathbb{Z}^3} |\mathcal{F}\phi(k)| \delta^{|k|} < +\infty, \tag{2.3}$$

where $\mathcal{F}\phi(k)$ is the k-th Fourier coefficient of ϕ defined by:

$$\mathcal{F}\phi(k) = \int_{\mathbb{T}^3} \phi(x) e^{-i2\pi k.x} dx.$$

The first theorem proves the existence of local analytic solutions of (1.7) with a life span uniform in ϵ.

Theorem 2.1. *Let $\delta_0 > 1$. Let $\rho_\epsilon(0)$ and $v_\epsilon(0)$ be two bounded families of B_{δ_0} such that $\int \rho_\epsilon(0) dx = 1$ and:*

$$\left| \int \rho_\epsilon(0) dx_\perp - 1 \right|_{\delta_0} \leq C\sqrt{\epsilon}, \tag{2.4}$$

where $C > 0$ is some given universal constant. Then there exists $\eta > 0$ such that for every $\delta_1 \in]1, \delta_0[$, for any $\epsilon > 0$, there exists a unique strong solution $(\rho_\epsilon, v_\epsilon)$ to (1.7) bounded uniformly in $\mathcal{C}([0, \eta(\delta_0 - \delta_1)[, B_{\delta_1})$ with initial conditions $(\rho_\epsilon(0), v_\epsilon(0))$. Moreover, $\sqrt{\epsilon}\partial_\parallel V_\epsilon$ is uniformly bounded in $\mathcal{C}([0, \eta(\delta_0 - \delta_1)[, B_{\delta_1})$.

Remark 2.1.
- *The condition (2.4) implies that $\sqrt{\epsilon}\partial_\parallel V_\epsilon(0)$ is bounded uniformly in B_{δ_0} (this is the correct scale in view of the energy conservation).*

- *Note that for all* $t \geq 0$, $\int \rho_\epsilon dx = 1$. *Hence the Poisson equation* $-\epsilon \partial_{\|}^2 V_\epsilon = \int \rho_\epsilon dx_\perp - 1$ *can always be solved.*
- *As explained in the introduction, due to the two-streams instabilities, we have to restrict to data with analytic regularity: the Sobolev version of these results is false in general (see [8] and the discussion of Section 5).*

We can then prove the convergence result:

Theorem 2.2. *Let* $(\rho_\epsilon, v_\epsilon)$ *be solutions to the system (1.7) for* $0 \leq t \leq T$ *satisfying for some* $s > 7/2$ *the following uniform estimates:*

$$(H) : \sup_{t \leq T, \epsilon} \left(\|\rho_\epsilon\|_{H^s_{x_\perp, x_\|}} + \|v_\epsilon\|_{H^s_{x_\perp, x_\|}} + \|\sqrt{\epsilon}\partial_{x_\|} V_\epsilon\|_{H^s_{x_\|}} \right) < +\infty. \tag{2.5}$$

Then, up to a subsequence, we get the following convergences

$$\rho_\epsilon \to \rho,$$

$$v_\epsilon - \frac{1}{i}(E_+ e^{it/\sqrt{\epsilon}} - E_- e^{-it/\sqrt{\epsilon}}) \to v,$$

strongly respectively in $\mathcal{C}([0,T], H^{s'}_{x_\perp, x_\|})$ *and* $\mathcal{C}([0,T], H^{s'-1}_{x_\perp, x_\|})$ *for all* $s' < s$, *and*

$$-\sqrt{\epsilon}\partial_{x_\|} V_\epsilon - (E_+ e^{it/\sqrt{\epsilon}} + E_- e^{-it/\sqrt{\epsilon}}) \to 0,$$

strongly in $\mathcal{C}([0,T], H^{s'}_{x_\|})$ *for all* $s' < s-1$, *and where* (ρ, v) *is solution to the asymptotic system (1.8) on* $[0,T]$ *with initial conditions:*

$$\rho(0) = \lim_{\epsilon \to 0} \rho_\epsilon(0),$$

$$v(0) = \lim_{\epsilon \to 0} \left(v_\epsilon(0) - \int \rho_\epsilon v_\epsilon dx_\perp(0) \right)$$

and $E_+(t, x_\|), E_-(t, x_\|)$ *are gradient correctors which satisfy the transport equations:*

$$\partial_t E_\pm + \left(\int \rho v dx_\perp \right) \partial_{x_\|} E_\pm = 0,$$

with initial data:

$$E_+(0) = \lim_{\epsilon \to 0} \frac{1}{2} \left(-\sqrt{\epsilon}\partial_{x_\|} V_\epsilon(0) + i \int \rho_\epsilon v_\epsilon dx_\perp(0) \right), \tag{2.6}$$

$$E_-(0) = \lim_{\epsilon \to 0} \frac{1}{2} \left(-\sqrt{\epsilon}\partial_{x_\|} V_\epsilon(0) - i \int \rho_\epsilon v_\epsilon dx_\perp(0) \right). \tag{2.7}$$

Remark 2.2.
- *It is clear that solutions built in Theorem 2.1 satisfy* (H).
- *If instead of* (H) *we make the stronger assumption, for* $\delta > 1$

$$(H') : \sup_{t \leq T, \epsilon} \left(\|\rho_\epsilon\|_{B_\delta} + \|v_\epsilon\|_{B_\delta} + \|\sqrt{\epsilon}\partial_{x_\|} V_\epsilon\|_{B_\delta} \right) < +\infty, \tag{2.8}$$

  (which is still satisfied by the solutions built in Theorem 2.1), then we get the same strong convergences in $\mathcal{C}([0,T], B_{\delta'})$ *for all* $\delta' < \delta$.

  Using Lemma 3.1 (ii), (iv), the proof under assumption (H') *is the same as under assumption* (H).

- *The "well-prepared" case corresponds to the case when:*

$$\lim_{\epsilon \to 0} -\sqrt{\epsilon}\partial_{x_\parallel} V_\epsilon(0) = 0,$$

$$\lim_{\epsilon \to 0} \int \rho_\epsilon v_\epsilon dx_\perp(0) = 0.$$

  Then there is no corrector.

With the same method used for Theorem 2.1, we can also prove a theorem of existence and uniqueness of analytic solutions to system (1.8).

Proposition 2.1. *Let $\delta_0 > 1$. For initial data $\rho(0), v(0) \in B_{\delta_0}$ satisfying*

$$\rho(0) \geq 0, \tag{2.9}$$

$$\int \rho(0) dx_\perp = 1 \tag{2.10}$$

and

$$\partial_\parallel \int \rho(0)v(0)dx_\perp = 0, \tag{2.11}$$

there exists $\eta > 0$ depending on δ_0 and on the initial conditions only such that there is a unique strong solution $(\rho, v_\parallel, p)$ to the system (1.8) with $\rho, v \in \mathcal{C}([0, \eta(\delta_0 - \delta_1)[, B_{\delta_1})$ for all $1 < \delta_1 < \delta_0$.

Remark 2.3. *The uniqueness proved in Proposition 2.1 allows to say that the convergences of Theorem 2.2 hold without having to consider subsequences, provided that the whole sequences of initial data converge to some functions in B_{δ_0} satisfying the assumptions of Proposition 2.1.*

3 Proof of Theorem 2.1

3.1 Functional analysis on B_δ spaces

First we define the time dependent analytic spaces we will work with.

Let β be an arbitrary constant in $]0, 1[$ (take for instance $\beta = 1/2$ to fix ideas) and $\eta > 0$ a parameter to be chosen later.

Definition. *Let $\delta_0 > 1$. We define the space $B^\eta_{\delta_0} = \{u \in \mathcal{C}^0([0, \eta(\delta_0 - 1)], B_{\delta_0 - t/\eta})\}$, endowed with the norm*

$$\|u\|_{\delta_0} = \sup_{\begin{cases} 1 < \delta \leq \delta_0 \\ 0 \leq t \leq \eta(\delta_0 - \delta) \end{cases}} \left(|u(t)|_\delta + \left(\delta_0 - \delta - \frac{t}{\eta}\right)^\beta |\nabla u(t)|_\delta \right),$$

where the norm $|u|_\delta$ was defined in (2.3):

$$|u|_\delta = \sum_{k \in \mathbb{Z}^3} |\mathcal{F}u(k)| \delta^{|k|},$$

We now gather from [14] a few elementary properties of these spaces, that we recall for the reader's convenience.

Lemma 3.1. *For all $\delta > 1$:*

(i) The spaces B_δ and B_δ^η are Banach algebra. More precisely, if $\phi_1, \phi_2 \in B_\delta$, and $\psi_1, \psi_2 \in B_\delta^\eta$ then:

$$|\phi_1 \phi_2|_\delta \leq |\phi_1|_\delta |\phi_2|_\delta,$$
$$\|\psi_1 \psi_2\|_\delta \leq \|\psi_1\|_\delta \|\psi_2\|_\delta.$$

(ii) If $\delta' < \delta$ then $B_\delta \subset B_{\delta'}$, the embedding being continuous and compact.

(iii) For all $s \in \mathbb{R}$, $B_\delta \subset H^s$, the embedding being continuous and compact.

(iv) For all $1 < \delta' < \delta$, if $\phi \in B_\delta$,

$$|\nabla \phi|_{\delta'} \leq \frac{\delta}{\delta - \delta'} |\phi|_\delta.$$

(v) If u is in $B_{\delta_0}^\eta$ and if $\delta' + t/\eta < \delta_0$ then

$$|\partial^2_{x_i,x_j} u(t)|_{\delta'} \leq 2\|u\|_{\delta_0} \delta_0 \left(\delta_0 - \delta' - \frac{t}{\eta} \right)^{-\beta-1}.$$

For further properties of these spaces we refer to the recent work of Mouhot and Villani [24], in which similar analytic spaces (and more sophisticated versions) are considered. The fact that considering analytic functions is useful both for the quasineutral limit (as studied here) and for the study of Landau damping (as done in [24]) is not a pure coincidence. Indeed, it turns out that because of scaling properties, these two questions are related (we refer for instance to the introduction of [19]).

Proof of Lemma 3.1. For the reader's convenience, we briefly sketch the proof (more details can be found in [14]). Point (i) can be readily checked from the Fourier series caracterization. We give an elementary proof for (ii) which is not given in [14] . The embedding is obvious. We consider for $N \in \mathbb{N}$ the map i_N defined by:

$$i_N(\phi) = \sum_{|k| \leq N} \mathcal{F}\phi(k) e^{i2\pi x.k}.$$

We then compute:

$$|(Id - i_N)\phi|_{\delta'} = \sum_{|k|>N} |\mathcal{F}\phi(k)| \delta'^{|k|} \leq \left(\frac{\delta'}{\delta}\right)^N \sum_{|k|>N} |\mathcal{F}\phi(k)| \delta^{|k|} \leq \left(\frac{\delta'}{\delta}\right)^N |\phi|_\delta.$$

So the embedding $B_\delta \subset B_{\delta'}$ is compact as the limit of finite rank operators. Point (iii) can be proved similarly. Point (iv) relies on the elementary estimate:

$$|k| \delta'^{|k|} \leq \frac{\delta}{\delta - \delta'} \delta^{|k|}.$$

For (v), consider $\delta = \delta' + \frac{\delta_0 - \delta' - t/\eta}{2}$ and apply (iv). □

We will also need the following elementary observation:

Remark 3.1. *Let $\phi \in B_\delta$. Then:*

$$\left| \int \phi \, dx_\perp \right|_\delta \leq |\phi|_\delta.$$

Proof. We simply compute:

$$\left|\int \phi dx_\perp\right|_\delta = \sum_{k=(0,k)\in\mathbb{N}^2\times\mathbb{N}} |\mathcal{F}(\phi)(k)|\delta^{|k|} \leq \sum_{k\in\mathbb{N}^3} |\mathcal{F}(\phi)|\delta^{|k|} = |\phi|_\delta.$$

□

3.2 Description of plasma oscillations

To simplify notations, we set $E_{\epsilon,\|} = -\partial_{x_\|} V_\epsilon(t, x_\|)$ (which has nothing to do with $E_\epsilon^\perp$). In this paragraph, we want to understand the oscillatory behaviour of $E_{\epsilon,\|}$. We will see that the dynamics in $x_\perp$ does not interfer too much with the equations on $E_{\epsilon,\|}$, so that we get almost the same description of oscillations as in Grenier's paper [14].

First we differentiate twice with respect to time the Poisson equation satisfied by V_ϵ:

$$\epsilon\partial_t^2\partial_{x_\|} E_{\epsilon,\|} = \partial_t^2 \int \rho_\epsilon dx_\perp. \tag{3.1}$$

Integrating with respect to $x_\perp$ the equation satisfied by ρ_ϵ, we obtain:

$$\partial_t \int \rho_\epsilon dx_\perp = \underbrace{-\int \nabla_\perp.(E_\epsilon^\perp \rho_\epsilon) dx_\perp}_{=0} - \partial_{x_\|} \int \rho_\epsilon v_\epsilon dx_\perp. \tag{3.2}$$

Then we integrate with respect to $x_\perp$ the equation satisfied by $\rho_\epsilon v_\epsilon$, that is:

$$\partial_t(\rho_\epsilon v_\epsilon) + \nabla_\perp.(E_\epsilon^\perp \rho_\epsilon v_\epsilon) + \partial_{x_\|}(v_\epsilon^2 \rho_\epsilon) = -\rho_\epsilon(\epsilon\partial_{x_\|}\phi_\epsilon(t,x) + \partial_{x_\|} V_\epsilon(t, x_\|))$$

and we get:

$$-\partial_t \int \rho_\epsilon v_\epsilon dx_\perp = \partial_{x_\|} \int \rho_\epsilon v_\epsilon^2 dx_\perp - E_{\epsilon,\|} \int \rho_\epsilon dx_\perp + \int \rho_\epsilon(\epsilon\partial_{x_\|}\phi_\epsilon) dx_\perp, \tag{3.3}$$

so that, combining (3.2) and (3.3):

$$\partial_t^2 \int \rho_\epsilon dx_\perp = \partial_{x_\|}^2 \int \rho_\epsilon v_\epsilon^2 dx_\perp - \partial_{x_\|}(E_{\epsilon,\|} \int \rho_\epsilon dx_\perp) + \partial_{x_\|} \int \rho_\epsilon(\epsilon\partial_{x_\|}\phi_\epsilon) dx_\perp. \tag{3.4}$$

Recall that by the Poisson equation:

$$\int \rho_\epsilon dx_\perp = 1 + \partial_{x_\|} E_{\epsilon,\|}.$$

Thus it comes by (3.1) and (3.4):

$$\epsilon\partial_t^2\partial_{x_\|} E_{\epsilon,\|} + \partial_{x_\|} E_{\epsilon,\|} = \partial_{x_\|}^2 \int \rho_\epsilon v_\epsilon^2 dx_\perp - \epsilon\partial_{x_\|}[E_{\epsilon,\|}\partial_{x_\|} E_{\epsilon,\|}] + \partial_{x_\|} \int \rho_\epsilon(\epsilon\partial_{x_\|}\phi_\epsilon) dx_\perp. \tag{3.5}$$

Equation (3.5) is the wave equation allowing to describe the essential oscillations. At least formally, this equation indicates that there are time oscillations with frequency $\frac{1}{\sqrt{\epsilon}}$ and magnitude $\frac{1}{\sqrt{\epsilon}}$ created by the right-hand side of the equation which acts like a source. We observe here that the source is expected to be of order $\mathcal{O}(1)$: indeed, by assumption on the data at $t = 0$, we can check that this quantity is bounded in a B_δ space.

In particular if we want to prove strong convergence results we will have to introduce non-trivial correctors in order to get rid of these oscillations. We notice also that (3.5) is very similar to the wave equation obtained in [14] (the only difference is a new term in the source), so that most of the calculations and estimates on $E_{\epsilon,\|}$ we will need are done in [14].

We have just observed that $E_{\epsilon,\|}$ roughly behaves like $\frac{1}{\sqrt{\epsilon}}e^{\pm it/\sqrt{\epsilon}}$. Hence if we consider the average in time:

$$G_\epsilon = \int_0^t E_{\epsilon,\|}(s, x_\|)ds, \tag{3.6}$$

we expect that G_ϵ is bounded uniformly with respect to ϵ in some functional space. We have the representation lemma which will be very useful to obtain a priori estimates:

Lemma 3.2. *The following identity holds:*

$$\mathcal{F}_\| G_\epsilon(t, k_\|) = \int_0^t \left(\frac{1}{ik_\|} \left[1 - \cos\left(\frac{t-s}{\sqrt{\epsilon}} \right) \right] \mathcal{F}_\| g_\epsilon(s, k_\|) \right) ds + \mathcal{F}_\| G_\epsilon^0, \tag{3.7}$$

denoting by $\mathcal{F}_\|$ the Fourier transform with respect to the parallel variable only and $k_\|$ the Fourier variable and where:

$$g_\epsilon = \partial_{x_\|}^2 \int \rho_\epsilon v_\epsilon^2 dx_\perp - \epsilon\partial_{x_\|}[E_{\epsilon,\|}\partial_{x_\|}E_{\epsilon,\|}] + \partial_{x_\|}\int \rho_\epsilon(\epsilon\partial_{x_\|}\phi_\epsilon)dx_\perp, \tag{3.8}$$

$$G_\epsilon^0 = \sqrt{\epsilon}E_{\epsilon,\|}(0, x_\|)\sin\left(\frac{s}{\sqrt{\epsilon}}\right) - \epsilon\partial_t E_{\epsilon,\|}(0, x_\|)\left(\cos\left(\frac{s}{\sqrt{\epsilon}}\right) - 1\right). \tag{3.9}$$

Proof of Lemma 3.2. We use Duhamel's formula for the "wave" equation (3.5) to get the following identity:

$$\mathcal{F}_\| E_\epsilon(t, k_\|) = \frac{1}{\sqrt{\epsilon}}\int_0^t \left(\frac{1}{ik_\|}\sin\left(\frac{t-s}{\sqrt{\epsilon}}\right)\mathcal{F}_\| g_\epsilon(s, k_\|) \right) ds + \mathcal{F}_\| E_\epsilon^0, \tag{3.10}$$

with g_ϵ defined in (3.8) and

$$E_{\epsilon,\|}^0 = E_{\epsilon,\|}(0, x)\cos(\frac{s}{\sqrt{\epsilon}}) + \sqrt{\epsilon}\partial_t E_{\epsilon,\|}(0, x)\sin(\frac{s}{\sqrt{\epsilon}}). \tag{3.11}$$

Then we can integrate this formula to recover (3.7). □

We now introduce the translated current (which corresponds to some filtering of the time oscillations created by the electric field):

$$w_\epsilon = v_\epsilon - G_\epsilon, \tag{3.12}$$

so that the transport equations of system (1.7) now read:

$$\begin{cases} \partial_t\rho_\epsilon + \nabla_\perp.(E_\epsilon^\perp\rho_\epsilon) + \partial_\|((w_\epsilon + G_\epsilon)\rho_\epsilon) = 0 \\ \partial_t w_\epsilon + \nabla_\perp.(E_\epsilon^\perp(w_\epsilon + G_\epsilon)) + (w_\epsilon + G_\epsilon)\partial_\|(w_\epsilon + G_\epsilon)) = -\epsilon\partial_\|\phi_\epsilon(t, x_\|). \end{cases} \tag{3.13}$$

3.3 Approximation scheme

To construct a solution, we use the usual approximation scheme for Cauchy-Kovalevskaya type of results ([7]). The principle is to define $\rho_\epsilon^n, w_\epsilon^n, G_\epsilon^n, V_\epsilon^n, \phi_\epsilon^n$ by recursion:

Initialization First of all, for $0 < t < \eta(\delta_0 - 1)$, $G_\epsilon^0(t)$ is given by formula (3.9); then for $0 < t < \eta(\delta_0 - 1)$, we can define:

$$\rho_\epsilon^0(t) = \rho_\epsilon(0),$$
$$w_\epsilon^0(t) = v_\epsilon(0) - G_\epsilon^0(t),$$
$$-\epsilon^2 \partial^2_{x_\parallel} \phi_\epsilon^0 - \Delta_{x_\perp} \phi_\epsilon^0 = \rho_\epsilon^0 - \int \rho_\epsilon^0 dx_\perp,$$
$$E_\epsilon^{\perp,0} = -\nabla^\perp \phi_\epsilon^0,$$

and $-\partial_{x_\parallel} V_\epsilon^0(t) = \partial_t G_\epsilon^0(t)$.

<u>Recursion</u> For $0 < t < \eta(\delta_0 - 1)$, we define $\rho_\epsilon^{n+1}, w_\epsilon^{n+1}$ by the transport equations:

$$\begin{cases} \partial_t \rho_\epsilon^{n+1} + \nabla_\perp.(E_\epsilon^{\perp,n}.\rho_\epsilon^n) + \partial_\parallel((w_\epsilon^n + G_\epsilon^n)\rho_\epsilon^n) = 0 \\ \partial_t w_\epsilon^{n+1} + \nabla_\perp.(E_\epsilon^{\perp,n}(w_\epsilon^n + G_\epsilon^n)) + (w_\epsilon^n + G_\epsilon^n)\partial_\parallel(w_\epsilon^n + G_\epsilon^n)) = -\epsilon \partial_\parallel \phi_\epsilon^n(t, x_\parallel), \end{cases} \tag{3.14}$$

with the initial conditions: $\rho_\epsilon^{n+1}(0) = \rho_\epsilon(0)$ and $w_\epsilon^{n+1} = v_\epsilon(0) - G_\epsilon^0$.
Then we can define ϕ_ϵ^{n+1} as the solution to the Poisson equation:

$$-\epsilon^2 \partial^2_{x_\parallel} \phi_\epsilon^{n+1} - \Delta_{x_\perp} \phi_\epsilon^{n+1} = \rho_\epsilon^{n+1} - \int \rho_\epsilon^{n+1} dx_\perp.$$

$$E_\epsilon^{\perp,n+1} = -\nabla^\perp \phi_\epsilon^{n+1},$$

Furthermore, we can define $G_\epsilon^{n+1}(t)$ by a variant of formula (3.7):

$$\mathcal{F}_\parallel G_\epsilon^{n+1}(t, k_\parallel) = \int_0^t \left(\frac{1}{ik_\parallel} \left[1 - \cos(\frac{t-s}{\sqrt{\epsilon}}) \right] \mathcal{F}_\parallel g_\epsilon^n(s, k_\parallel) \right) ds + \mathcal{F}_\parallel G_\epsilon^0, \tag{3.15}$$

with $g_\epsilon^n = \partial^2_{x_\parallel} \int \rho_\epsilon^n (w_\epsilon^n + G_\epsilon^n)^2 dx_\perp - \epsilon \partial_{x_\parallel}[E_{\epsilon,\parallel}^n \partial_{x_\parallel} E_{\epsilon,\parallel}^n] + \partial_{x_\parallel} \int \rho_\epsilon^n (\epsilon \partial_{x_\parallel} \phi_\epsilon^n) dx_\perp$.
Finally we define:

$$-\epsilon \partial_{x_\parallel} V_\epsilon^{n+1} = \partial_t G^{n+1}(t).$$

3.4 A priori estimates

Let $n \geq 0$. The goal is now to prove some a priori estimates for $G_\epsilon^{n+1}, \rho_\epsilon^{n+1}$ and w_ϵ^{n+1} (in terms of $G_\epsilon^n, \rho_\epsilon^n$ and w_ϵ^n). We are also able to get similar estimates on $E_\epsilon^{\perp,n+1}$ and $\epsilon \partial_{x_\parallel} \phi_\epsilon^{n+1}$, thanks to the Poisson equation satisfied by ϕ_ϵ^{n+1}. Ultimately the goal is to prove that if the parameter η is chosen small enough, then all these sequences are Cauchy sequences in $B^\eta_{\delta_0}$.

3.4.1 Estimate on G_ϵ^{n+1} and $\sqrt{\epsilon} E_{\epsilon,\parallel}^{n+1}$

The first aim in this paragraph is to estimate $\|G_\epsilon^{n+1}\|_{\delta_0}$, using (3.15). We have:

$$\begin{aligned} |G_\epsilon^{n+1}|_\delta &\leq \left| \int_0^t \mathcal{F}_\parallel^{-1} \left(\frac{1}{ik_\parallel} [1 - \cos(\frac{t-s}{\sqrt{\epsilon}})] \mathcal{F}_\parallel g_\epsilon^n(s, k_\parallel) \right) ds \right|_\delta + |G_\epsilon^0|_\delta \\ &\leq 2 \int_0^t \left| \mathcal{F}_\parallel^{-1} \left(\frac{1}{ik_\parallel} \mathcal{F}_\parallel g_\epsilon^n(s, k_\parallel) \right) \right|_\delta ds + |G_\epsilon^0|_\delta, \end{aligned}$$

with:

$$\frac{1}{ik_\parallel}\mathcal{F}_\parallel g_\epsilon^n = \mathcal{F}_\parallel\left(\partial_{x_\parallel}\int \rho_\epsilon(w_\epsilon^n + G_\epsilon^n)^2 dx_\perp\right) - \epsilon\mathcal{F}_\parallel\left(E^n_{\epsilon,\parallel}\partial_{x_\parallel}E^n_{\epsilon,\parallel}\right) + \mathcal{F}_\parallel\left(\left(\int \rho_\epsilon^n(\epsilon\partial_{x_\parallel}\phi_\epsilon^n)dx_\perp\right)\right).$$

Thanks to Remark 3.1 and Lemma 3.1 , (i), we first estimate:

$$\begin{aligned}\left|\int \partial_{x_\parallel}(\rho_\epsilon^n(w_\epsilon^n + G_\epsilon^n)^2)dx_\perp\right|_\delta \le \left|\partial_{x_\parallel}(\rho_\epsilon^n(w_\epsilon^n + G_\epsilon^n)^2)\right|_\delta \le &(\delta_0-\delta-\frac{s}{\eta})^{-\beta}\|\rho_\epsilon^n(w_\epsilon^n + G_\epsilon^n)^2\|_{\delta_0}\\ \le&(\delta_0-\delta-\frac{s}{\eta})^{-\beta}\|\rho_\epsilon^n\|_{\delta_0}\|w_\epsilon^n + G_\epsilon^n\|^2_{\delta_0}.\end{aligned} \tag{3.16}$$

Similarly, we prove:

$$\begin{aligned}\epsilon\left|E^n_{\epsilon,\parallel}\partial_{x_\parallel}E^n_{\epsilon,\parallel}\right|_\delta &\le \frac{1}{2}\left|\partial_{x_\parallel}(\sqrt{\epsilon}E^n_{\epsilon,\parallel})^2\right|_\delta\\ &\le \frac{1}{2}(\delta_0-\delta-\frac{s}{\eta})^{-\beta}\left\|(\sqrt{\epsilon}E^n_{\epsilon,\parallel})^2\right\|_{\delta_0}\\ &\le \frac{1}{2}(\delta_0-\delta-\frac{s}{\eta})^{-\beta}\|\sqrt{\epsilon}E^n_{\epsilon,\parallel}\|^2_{\delta_0},\end{aligned} \tag{3.17}$$

$$\left|\int \partial_{x_\parallel}\left(\rho_\epsilon^n(\epsilon\partial_{x_\parallel}\phi_\epsilon^n)\right)dx_\perp\right|_\delta \le (\delta_0-\delta-\frac{s}{\eta})^{-\beta}\|\rho_\epsilon^n\|_{\delta_0}\|\epsilon\partial_{x_\parallel}\phi_\epsilon^n\|_{\delta_0}.$$

Thus, we finally obtain:

$$|G_\epsilon^{n+1}|_\delta \le 2\int_0^t(\delta_0-\delta-\frac{s}{\eta})^{(-\beta)}(\|\rho_\epsilon\|_{\delta_0}\|w_\epsilon^n + G_\epsilon^n\|^2_{\delta_0} + \|\sqrt{\epsilon}E^n_{\epsilon,\parallel}\|^2_{\delta_0} + \|\rho_\epsilon^n\|_{\delta_0}\|\epsilon\partial_{x_\parallel}\phi_\epsilon^n\|_{\delta_0})ds + |G_\epsilon^0|_\delta.$$

In what follows, $C(\delta_0,\beta)$ is a constant depending only on δ_0 and β that may change from one line to another. As before, one can show (this time we use lemma 3.1, (v)) that:

$$\begin{aligned}|\partial_{x_\parallel}G_\epsilon^{n+1}|_\delta \le C(\delta_0,\beta)\textstyle\int_0^t(\delta_0-\delta-\frac{s}{\eta})^{(-\beta-1)}&\Big(\|\rho_\epsilon^n\|_{\delta_0}\|w_\epsilon^n + G_\epsilon^n\|^2_{\delta_0} + \|\sqrt{\epsilon}E^n_{\epsilon,\parallel}\|^2_{\delta_0}\\ &+\|\rho_\epsilon^n\|_{\delta_0}\|\epsilon\partial_{x_\parallel}\phi_\epsilon^n\|_{\delta_0}\Big)ds + |\partial_{x_\parallel}G_\epsilon^0|_\delta.\end{aligned}$$

Hence using the elementary estimates

$$\int_0^t \frac{ds}{(\delta_0-\delta-\frac{s}{\eta})^\beta} \le \eta\frac{2}{1-\beta}\delta_0^{1-\beta},$$

$$\int_0^t \frac{ds}{(\delta_0-\delta-\frac{s}{\eta})^{\beta+1}} \le \frac{2\eta}{\beta}(\delta_0-\delta-\frac{t}{\eta})^{-\beta},$$

we get:

$$\|G_\epsilon^{n+1}\|_{\delta_0} \le \eta C(\delta_0,\beta)\left((\|w_\epsilon^n\|_{\delta_0} + \|G_\epsilon^n\|_{\delta_0})^2\|\rho_\epsilon^n\|_{\delta_0} + \|\sqrt{\epsilon}E^n_{\epsilon,\parallel}\|^2_{\delta_0} + \|\rho_\epsilon^n\|_{\delta_0}\|\epsilon\partial_{x_\parallel}\phi_\epsilon^n\|_{\delta_0}\right) + \|G_\epsilon^0\|_{\delta_0}. \tag{3.18}$$

Finally, we compare two solutions $(w_\epsilon^{n+1},\rho_\epsilon^{n+1},G_\epsilon^{n+1})$ and $(w_\epsilon^{n+2},\rho_\epsilon^{n+2},G_\epsilon^{n+2})$ (observe that these have the same initial data).

$$|G_\epsilon^{n+2} - G_\epsilon^{n+1}|_\delta \le \int_0^t\left|\mathcal{F}_\parallel^{-1}\left(\frac{1}{ik_\parallel}[1-\cos(\frac{t-s}{\sqrt{\epsilon}})]\left[\mathcal{F}_\parallel g_\epsilon^{n+1}(s,k_\parallel) - \mathcal{F}_\parallel g_\epsilon^n(s,k_\parallel)\right]\right)\right|_\delta ds, \tag{3.19}$$

We decompose the products appearing in $g_\epsilon^{n+1} - g_\epsilon^n$ in the following way:

$$\rho_\epsilon^{n+1}(w_\epsilon^{n+1})^2 - \rho_\epsilon^n(w_\epsilon^n)^2 = (\rho_\epsilon^{n+1} - \rho_\epsilon^n)(w_\epsilon^{n+1})^2 + (w_\epsilon^{n+1} - w_\epsilon^n)(w_\epsilon^{n+1} + w_\epsilon^n)\rho_\epsilon^n,$$

and we proceed likewise for the other terms. Then we obtain the following estimate with the same method as before:

$$\begin{aligned}
\|G_\epsilon^{n+1} - G_\epsilon^{n+2}\|_{\delta_0} \quad \leq \quad & \eta C(\delta_0,\beta)\Big((\|w_\epsilon^{n+1} - w_\epsilon^n\|_{\delta_0} + \|G_\epsilon^{n+1} - G_\epsilon^n\|_{\delta_0}) \\
& \times(\|w_\epsilon^{n+1}\|_{\delta_0} + \|w_\epsilon^n\|_{\delta_0} + \|G_\epsilon^{n+1}\|_{\delta_0} + \|G_\epsilon^n\|_{\delta_0})(\|\rho_\epsilon^{n+1}\|_{\delta_0} + \|\rho_\epsilon^n\|_{\delta_0}) \\
+ \quad & \|\rho_\epsilon^{n+1} - \rho_\epsilon^n\|_{\delta_0}(\|w_\epsilon^{n+1}\|_{\delta_0}^2 + \|w_\epsilon^n\|_{\delta_0}^2 + \|G_\epsilon^{n+1}\|_{\delta_0}^2 + \|G_\epsilon^n\|_{\delta_0}^2) \\
+ \quad & \|\rho_\epsilon^{n+1} - \rho_\epsilon^n\|_{\delta_0}(\|\epsilon\partial_{x_\parallel}\phi_\epsilon^{n+1}\|_{\delta_0} + \|\epsilon\partial_{x_\parallel}\phi_\epsilon^n\|_{\delta_0}) \\
+ \quad & \|\epsilon\partial_{x_\parallel}\phi_\epsilon^{n+1} - \epsilon\partial_{x_\parallel}\phi_\epsilon^n\|_{\delta_0}(\|\rho_\epsilon^{n+1}\|_{\delta_0} + \|\rho_\epsilon^n\|_{\delta_0}) \\
+ \quad & \|\sqrt{\epsilon}E_{\epsilon,\parallel}^{n+1} - \sqrt{\epsilon}E_{\epsilon,\parallel}^n\|_{\delta_0}(\|\sqrt{\epsilon}E_{\epsilon,\parallel}^{n+1}\|_{\delta_0} + \|\sqrt{\epsilon}E_{\epsilon,\parallel}^n\|_{\delta_0})\Big).
\end{aligned} \tag{3.20}$$

Likewise we get the same kind of estimates for $\|\sqrt{\epsilon}E_{\epsilon,\parallel}^{n+1}\|_{\delta_0}$ since from (3.10) we have the formula:

$$\mathcal{F}_\parallel(\sqrt{\epsilon}E_{\epsilon,\parallel}^{n+1})(t,k_\parallel) = \int_0^t \left(\frac{1}{ik_\parallel}[\sin(\frac{t-s}{\sqrt{\epsilon}})]\mathcal{F}_\parallel g_\epsilon^n(s,k_\parallel)\right) ds + \mathcal{F}_\parallel(\sqrt{\epsilon}E_{\epsilon,\parallel}^0), \tag{3.21}$$

3.4.2 Estimate on $E_\epsilon^{\perp,n+1}$ and $\epsilon\partial_{x_\parallel}\phi_\epsilon^{n+1}$

We now use the scaled Poisson equation satisfied by ϕ_ϵ^{n+1} to get some similar a priori estimates. For the reader's convenience, we first recall this equation:

$$-\epsilon^2\partial_{x_\parallel}^2\phi_\epsilon^{n+1} - \Delta_\perp\phi_\epsilon^{n+1} = \rho_\epsilon^{n+1} - \int \rho_\epsilon^{n+1}dx_\perp.$$

The principle here is to look at the symbols of the operators involved in the Poisson equation. Accordingly, we compute in Fourier variables:

$$\epsilon^2k_\parallel^2\mathcal{F}\phi_\epsilon^{n+1} + |k_\perp|^2\mathcal{F}\phi_\epsilon^{n+1} = \mathcal{F}\left(\rho_\epsilon^{n+1} - \int \rho_\epsilon^{n+1}dx_\perp\right). \tag{3.22}$$

Thus it comes:

$$\mathcal{F}\phi_\epsilon^{n+1} = \frac{\mathcal{F}(\rho_\epsilon^{n+1} - \int \rho_\epsilon^{n+1}dx_\perp)}{\epsilon^2k_\parallel^2 + |k_\perp|^2}.$$

Since $\int(\rho_\epsilon^{n+1} - \int \rho_\epsilon^{n+1}dx_\perp)dx_\perp = 0$, we have for all $k_\parallel \in \mathbb{Z}$:

$$\mathcal{F}\left(\rho_\epsilon^{n+1} - \int \rho_\epsilon^{n+1}dx_\perp\right)(0,k_\parallel) = 0.$$

Thus we get, for all $k_\perp, k_\parallel \in \mathbb{Z}$:

$$|\mathcal{F}\phi_\epsilon^{n+1}| \leq \frac{|\mathcal{F}(\rho_\epsilon^{n+1} - \int \rho_\epsilon^{n+1}dx_\perp)|}{|k_\perp|^2}.$$

In particular we easily get, using the relation $E_\epsilon^{\perp,n+1} = -\nabla^\perp\phi_\epsilon^{n+1}$:

$$|\mathcal{F}E_\epsilon^{\perp,n+1}| \le \frac{|\mathcal{F}(\rho_\epsilon^{n+1} - \int \rho_\epsilon^{n+1} dx_\perp)|}{|k_\perp|} \le \left|\mathcal{F}\left(\rho_\epsilon^{n+1} - \int \rho_\epsilon^{n+1} dx_\perp\right)\right|.$$

Hence:

$$\|E_\epsilon^{\perp,n+1}\|_{\delta_0} \le 2\|\rho_\epsilon^{n+1}\|_{\delta_0}. \tag{3.23}$$

Likewise, using the elementary inequality $ab \le \frac{1}{2}(a^2+b^2)$ and $|k_\perp| \ge 1$:

$$|\mathcal{F}(\epsilon\partial_{x_\parallel}\phi_\epsilon^{n+1})| \le \frac{\epsilon|k_\parallel||\mathcal{F}(\rho_\epsilon - \int \rho_\epsilon dx_\perp)|}{\epsilon^2 k_\parallel^2 + |k_\perp|^2} \le \frac{1}{2}|\mathcal{F}(\rho_\epsilon^{n+1} - \int \rho_\epsilon^{n+1} dx_\perp)|,$$

and consequently:

$$\|\epsilon\partial_{x_\parallel}\phi_\epsilon^{n+1}\|_{\delta_0} \le \|\rho_\epsilon^{n+1}\|_{\delta_0}. \tag{3.24}$$

Finally, if we compare two solutions at step $n+1$ and $n+2$:

$$\|E_\epsilon^{\perp,n+2} - E_\epsilon^{\perp,n+1}\|_{\delta_0} + \|\epsilon\partial_{x_\parallel}\phi_\epsilon^{n+2} - \epsilon\partial_{x_\parallel}\phi_\epsilon^{n+1}\|_{\delta_0} \le 2\|\rho_\epsilon^{n+2} - \rho_\epsilon^{n+1}\|_{\delta_0}. \tag{3.25}$$

3.4.3 Estimate on ρ_ϵ^{n+1} and w_ϵ^{n+1}

We now use the conservation laws satisfied by ρ_ϵ^{n+1} and w_ϵ^{n+1} to get the appropriate estimates. We first recall that the density ρ_ϵ^{n+1} satisfies the equation:

$$\partial_t\rho_\epsilon^{n+1} + \nabla_\perp.(E_\epsilon^{\perp,n}\rho_\epsilon^n) + \partial_\parallel((w_\epsilon^n + G_\epsilon^n)\rho_\epsilon^n) = 0.$$

Writing $\rho_\epsilon^{n+1} = \int_0^t \partial_t\rho_\epsilon^{n+1} ds + \rho_\epsilon(0)$, we get:

$$|\rho_\epsilon^{n+1}|_\delta \quad \le \quad \int_0^t |\partial_t\rho_\epsilon^{n+1}|_\delta ds + |\rho_\epsilon(0)|_\delta$$

With the same kind of computations as before and using estimate (3.23) we get:

$$|\nabla_\perp.(E_\epsilon^{\perp,n}\rho_\epsilon^n)|_\delta \le (\delta_0 - \delta - \frac{s}{\eta})^{-\beta}\|E_\epsilon^{\perp,n}\|_{\delta_0}\|\rho_\epsilon^n\|_{\delta_0} \le 2(\delta_0 - \delta - \frac{s}{\eta})^{-\beta}\|\rho_\epsilon^n\|_{\delta_0}^2,$$

$$|\partial_\parallel((w_\epsilon^n + G_\epsilon^n)\rho_\epsilon)|_\delta \le (\delta_0 - \delta - \frac{s}{\eta})^{-\beta}\|w_\epsilon^n + G_\epsilon^n)\|_{\delta_0}\|\rho_\epsilon^n\|_{\delta_0}.$$

As a consequence we obtain:

$$|\rho_\epsilon^{n+1}|_\delta \le |\rho_\epsilon(0)|_\delta + C(\delta_0,\beta)\int_0^t (\delta_0 - \delta - \frac{s}{\eta})^{-\beta}\|\rho_\epsilon^n\|_{\delta_0}(\|\rho_\epsilon^n\|_{\delta_0} + \|w_\epsilon^n\|_{\delta_0} + \|G_\epsilon^n\|_{\delta_0})ds.$$

Similarly we estimate $|\partial_{x_i}\rho_\epsilon^{n+1}|_\delta$ by differentiating with respect to x_i the equation satisfied by ρ_ϵ^{n+1}. Finally we get:

$$\|\rho_\epsilon^{n+1}\|_{\delta_0} \le \eta C(\delta_0,\beta)\|\rho_\epsilon^n\|_{\delta_0}(\|\rho_\epsilon^n\|_{\delta_0} + \|w_\epsilon^n\|_{\delta_0} + \|G_\epsilon^n\|_{\delta_0}) + \|\nabla\rho_\epsilon(0)\|_{\delta_0}. \tag{3.26}$$

If we compare solutions at steps $n+1$ and $n+2$, we get likewise:

$$\begin{aligned}\|\rho_\epsilon^{n+2} - \rho_\epsilon^{n+1}\|_{\delta_0} \quad &\le \quad \eta C(\delta_0,\beta)\Big((\|\rho_\epsilon^{n+1}\|_{\delta_0} + \|\rho_\epsilon^n\|_{\delta_0})(\|w_\epsilon^{n+1} - w_\epsilon^n\|_{\delta_0} + \|G_\epsilon^{n+1} - G_\epsilon^n\|_{\delta_0}) \\ &+ \quad (\|\rho_\epsilon^{n+1}\|_{\delta_0} + \|\rho_\epsilon^n\|_{\delta_0} + \|w_\epsilon^{n+1}\|_{\delta_0} + \|w_\epsilon^n\|_{\delta_0} + \|G_\epsilon^{n+1}\|_{\delta_0} + \|G_\epsilon^n\|_{\delta_0}) \\ &\times \quad (\|\rho_\epsilon^{n+1} - \rho_\epsilon^n\|_{\delta_0})\Big).\end{aligned} \tag{3.27}$$

In the same fashion, we recall that w_ϵ^{n+1} satisfies the following transport equation:

$$\partial_t w_\epsilon^{n+1} + \nabla_\perp.(E_\epsilon^{\perp,n}(w_\epsilon^n + G_\epsilon^n)) + (w_\epsilon^n + G_\epsilon^n)\partial_\|(w_\epsilon^n + G_\epsilon^n)) = -\epsilon\partial_\|\phi_\epsilon^n(t,x_\|),$$

and we can once again estimate the δ_0 norm of w_ϵ^{n+1}:

$$\|w_\epsilon^{n+1}\|_{\delta_0} \leq \eta C(\delta_0,\beta)\left((\|w_\epsilon^n\|_{\delta_0} + \|G_\epsilon^n\|_{\delta_0})\|\rho_\epsilon^n\|_{\delta_0} + (\|w_\epsilon^n\|_{\delta_0} + \|G_\epsilon^n\|_{\delta_0})^2 + \|\epsilon\partial_\|\phi_\epsilon^n\|_{\delta_0}\right), \tag{3.28}$$

and if we compare two solutions at steps $n+1$ and $n+2$:

$$\begin{aligned}\|w_\epsilon^{n+2} - w_\epsilon^{n+1}\|_{\delta_0} \quad \leq \quad & \eta C(\delta_0,\beta)\Big((\|\rho_\epsilon^{n+1}\|_{\delta_0} + \|\rho_\epsilon^n\|_{\delta_0})(\|w_\epsilon^{n+1} - w_\epsilon^n\|_{\delta_0} + \|G_\epsilon^{n+1} - G_\epsilon^n\|_{\delta_0}) \\ + \quad & (\|w_\epsilon^{n+1}\|_{\delta_0} + \|w_\epsilon^n\|_{\delta_0} + \|G_\epsilon^{n+1}\|_{\delta_0} + \|G_\epsilon^n\|_{\delta_0}) \\ & \times(\|w_\epsilon^{n+1} - w_\epsilon^n\|_{\delta_0} + \|G_\epsilon^{n+1} - G_\epsilon^n\|_{\delta_0} + \|\rho_\epsilon^{n+1} - \rho_\epsilon^n\|_{\delta_0}) \\ & \|\epsilon\partial_\|\phi_\epsilon^{n+1} - \epsilon\partial_\|\phi_\epsilon^{n+1}\|_{\delta_0}\Big).\end{aligned} \tag{3.29}$$

3.5 Finding a fixed point

We are now in position to use our estimates to prove the existence and uniqueness of a fixed point.

First let C_1 defined by:

$$C_1 = \sup_{\eta\leq 1}\left\{\|\rho_\epsilon(0)\|_{\delta_0}, \|w_\epsilon(0)\|_{\delta_0}, \|G_\epsilon(0)\|_{\delta_0}, \|\sqrt{\epsilon}E_\epsilon(0)\|_{\delta_0}, 1\right\}$$

Let $C_2 = C_1 + 1$. It is possible to choose η small enough with respect to C_1 to propagate the following estimates by recursion (we refer to [14] for more details; more explicitly $\eta = \frac{1}{200C(\delta_0,\beta)C_2^3)}$ is for instance convenient). At Step n ($n \geq 1$), the property reads:

(i)

$$\begin{cases} \|\rho_\epsilon^n\|_{\delta_0} \leq C_2, \\ \|w_\epsilon^n\|_{\delta_0} \leq C_2, \\ \|G_\epsilon^n\|_{\delta_0} \leq C_2, \\ \|\sqrt{\epsilon}E_{\epsilon,\|}^n\|_{\delta_0} \leq C_2. \end{cases}$$

(ii)

$$\begin{cases} \|\rho_\epsilon^n - \rho_\epsilon^{n-1}\|_{\delta_0} \leq \frac{C_2}{2^n}, \\ \|w_\epsilon^n - w_\epsilon^{n-1}\|_{\delta_0} \leq \frac{C_2}{2^n}, \\ \|G_\epsilon^n - G_\epsilon^{n-1}\|_{\delta_0} \leq \frac{C_2}{2^n}, \\ \|\sqrt{\epsilon}E_{\epsilon,\|}^n - \sqrt{\epsilon}E_{\epsilon,\|}^{n-1}\|_{\delta_0} \leq \frac{C_2}{2^n}. \end{cases}$$

One first checks that (i) is satisfied for $n = 0$. In particular for the last condition, we use (2.4). As in [14], checking that (ii) is satisfied for $n = 1$ in fact needs a special treatment which is very similar to the general case, so we will not detail it.

To propagate these estimates for $n \geq 1$, we use the crucial estimates (3.20),(3.27),(3.29). Let us briefly explain the passage from Step $(n+1)$ to Step $(n+2)$ by examining the case of Property (ii) for G_ϵ^n (the other cases are treated similarly). Using (3.20) and the Properties (i) and (ii) at step $n+1$ we have:

$$\|G_\epsilon^{n+1} - G_\epsilon^{n+2}\|_{\delta_0} \leq \eta C(\delta_0,\beta)\frac{C_2}{2^{n+1}}30C_2,$$

and with our choice of η, we notice that $\eta C(\delta_0,\beta)\frac{C_2}{2^{n+1}}30C_2^2 \leq \frac{C_2}{2^{n+2}}$, which proves the property (ii) for G_ϵ at step $(n+2)$.

This proves that the sequences $\rho_\epsilon^n, w_\epsilon^n, G_\epsilon^n, \sqrt{\epsilon}E_\epsilon, E_\epsilon^{\perp,n}, \epsilon\partial_{x_\parallel}\phi_\epsilon^n$ are Cauchy sequences (with respect to n) in $B_{\delta_0}^\eta$, and consequently converge strongly in $B_{\delta_0}^\eta$, the estimates being uniform in ϵ. It is clear that the limit satisfies System (1.7). The requirement $\delta_1 < \delta_0$ and the explicit life span in Theorem 2.1 come directly from the definition of the $B_{\delta_0}^\eta$ spaces.

For the uniqueness part, one can simply notice that the estimates we have shown allow us to prove that the application $\mathfrak{F}$ defined by:

$$\mathfrak{F}(\rho_\epsilon, w_\epsilon) = \begin{pmatrix} \int_0^t (-\nabla_\perp.(E_\epsilon^\perp \rho_\epsilon) - \partial_\parallel((w_\epsilon + G_\epsilon)\rho_\epsilon))ds \\ \int_0^t (-\nabla_\perp.(E_\epsilon^\perp(w_\epsilon + G_\epsilon)) - (w_\epsilon + G_\epsilon)\partial_\parallel(w_\epsilon + G_\epsilon)) - \epsilon\partial_\parallel \phi_\epsilon(t, x_\parallel))ds \end{pmatrix},$$

is a contraction on the closed subset B of $B_{\delta_0} \times B_{\delta_0}$, defined by:

$$B = \{\rho, w \in B_{\delta_0}; \|\rho\|_{\delta_0} \leq C, \|w\|_{\delta_0} \leq C\},$$

with C large enough, provided that η is chosen small enough. The uniqueness of the analytic solution then follows.

3.6 Proof of Proposition 2.1

We can lead the same analysis as for the proof of Theorem 2.1, but even simpler since here we do not have to deal anymore with the fast oscillations in time. The only slightly different point is to estimate the norm of $\int_0^t -\partial_\parallel p ds = \int_0^t \partial_\parallel \int \rho v^2 dx_\perp ds$, which is straightforward:

$$\left\| \int_0^t \partial_\parallel p ds \right\|_{\delta_0} \leq \eta C \|\rho\|_{\delta_0} \|v\|_{\delta_0}^2.$$

Then as before, we can use a contraction argument to prove the proposition.

4 Proof of Theorem 2.2

Step 1: Another average in time for $E_{\epsilon,\parallel}$

We have observed previously that the wave equation (3.5) describing the time oscillations of $E_{\epsilon,\parallel}$ was the same as the one appearing in Grenier's work, except for a slight change in the source. Therefore the following decomposition taken from [14, Proposition 3.1.1] identically holds, since the proof only relies on the fact that the source g_ϵ (defined in (3.8)) is bounded in $L_t^\infty H_x^{s-1}$, which is still the case here, under the assumptions of Theorem (2.2).

Lemma 4.1. *Under assumption (H), there exist $E_\epsilon^{(1)}, E_\epsilon^{(2)}$ and W_ϵ such that $E_{\epsilon,\parallel} = E_\epsilon^{(1)} + E_\epsilon^{(2)}$ and a positive constant C independent of ϵ such as:*

(i) $\|\sqrt{\epsilon}E_\epsilon^{(1)}\|_{L^\infty(H_{x_\parallel}^{s-1})} \leq C$.

(ii) $\partial_t W_\epsilon = E_\epsilon^{(1)}$, $\|W_\epsilon\|_{L^\infty(H_{x_\parallel}^{s-1})} \leq C$ *and* $W_\epsilon \rightharpoonup 0$ *in* L^2.

(iii) $W_\epsilon(0) = -\epsilon\partial_t E_{\epsilon,\parallel}(0) = \int \rho_\epsilon(0)v_\epsilon(0)dx_\perp$.

(iv) $\|E_\epsilon^{(2)}\|_{L^\infty(H_{x_\parallel}^{s-1})} \leq C$.

(v) $\int E_\epsilon^{(1)} dx_\parallel = \int E_\epsilon^{(2)} dx_\parallel = 0$.

Roughly speaking, this lemma allows to decompose $E_{\epsilon,\parallel}$ into a oscillating part with magnitude $\frac{1}{\sqrt{\epsilon}}$ that we will have to filter out and a bounded part that will give rise to the pressure term.

Step 2: Uniform bound on $E_\epsilon^\perp$ and $\partial_{x_\parallel}\phi_\epsilon$

Under hypothesis (H), using the Poisson equation satisfied by ϕ_ϵ, one can check that $E_\epsilon^\perp$ and $\partial_{x_\parallel}\phi_\epsilon$ are bounded in $L^\infty_t(H^{s-1})$ uniformly with respect to ϵ (we do not need any gain of elliptic regularity). Indeed, since:

$$\int \left(\rho_\epsilon - \int \rho_\epsilon dx_\perp\right) dx_\perp = 0,$$

we can use the trivial bound on the symbol

$$\frac{1}{|k_\perp|^2 + \epsilon^2 |k_\parallel|^2} \leq 1, \quad \text{for } k_\perp \neq 0$$

to get

$$\|\phi_\epsilon\|_{H^s_{x_\perp, x_\parallel}} \leq \left\| \rho - \int \rho dx_\perp \right\|_{H^s_{x_\perp, x_\parallel}}.$$

Hence the result holds.

Step 3: Passage to the limit

Let $w_\epsilon = v_\epsilon - W_\epsilon$. According to assumption (H) and Lemma 4.1, w_ϵ is uniformly bounded in $L^\infty_t([0,T], H^{s-1})$. On the other hand, we have :

$$\partial_t w_\epsilon + \nabla_\perp.(E_\epsilon^\perp w_\epsilon) + w_\epsilon \partial_{x_\parallel} w_\epsilon = -\epsilon \partial_{x_\parallel}\phi_\epsilon + E_\epsilon^{(2)} - w_\epsilon \partial_{x_\parallel} W_\epsilon - W_\epsilon \partial_{x_\parallel} w_\epsilon - W_\epsilon \partial_{x_\parallel} W_\epsilon. \quad (4.1)$$

(Notice that $\nabla_\perp.(E_\epsilon^\perp W_\epsilon) = W_\epsilon \nabla_\perp.(E_\epsilon^\perp) = 0$.)

Thus, using the uniform bounds of assumption (H) and the fact the H^{s-2}_x is an algebra, we can easily see that $\partial_t w_\epsilon$ is bounded in $L^\infty_t([0,T], H^{s-2})$. Thanks to the Aubin-Lions lemma (see for instance [26]), w_ϵ converges strongly (up to a subsequence) to some function w in $\mathcal{C}([0,T], H^{s'-1})$ for all $s' < s$.

According to Step 2, $\epsilon \partial_{x_\parallel}\phi_\epsilon \rightharpoonup 0$ in the distributional sense.

Since w_ϵ strongly converges in $\mathcal{C}([0,T], H^{s'-1})$, it also converges strongly in $L^2([0,T], L^2)$ and by Lemma 4.1, (ii), W_ϵ weakly converges to 0 in $L^2([0,T], L^2)$. Thus, the following convergence also holds in the sense of distributions:

$$-w_\epsilon \partial_{x_\parallel} W_\epsilon - W_\epsilon \partial_{x_\parallel} w_\epsilon \rightharpoonup 0,$$

and $-W_\epsilon \partial_{x_\parallel} W_\epsilon + E_\epsilon^{(2)}$ weakly converges (up to a subsequence) to some function F since this term is uniformly bounded in $L^\infty([0,T], H^{s-2}_{x_\parallel})$.

Furthermore, we observe that:

$$\int \left(-W_\epsilon \partial_{x_\parallel} W_\epsilon + E_\epsilon^{(2)}\right) dx_\parallel = \int \left(-\frac{1}{2}\partial_{x_\parallel} W_\epsilon^2 + E_\epsilon^{(2)}\right) dx_\parallel = 0,$$

using Lemma 4.1, (v). This implies that $\int F dx_\parallel = 0$, and thus there exists p such that $F = -\partial_{x_\parallel} p$.

Since $E_\epsilon^\perp$ is uniformly bounded in $L_t^\infty([0,T],H^{s-1})$, it also weakly converges, up to a subsequence, to some function $E^\perp$.

We now use the strong limit of w_ϵ in $\mathcal{C}([0,T],H^{s'-1})$ in order to pass to the limit in the sense of distributions in the convection terms. As a consequence, we obtain, passing to the limit in the sense of distributions:

$$\partial_t w + \nabla_\perp.(E^\perp w) + w\partial_{x_\parallel} w = -\partial_{x_\parallel} p. \tag{4.2}$$

We recall now that the equation satisfied by ρ_ϵ is:

$$\partial_t \rho_\epsilon + \nabla_\perp.(E_\epsilon^\perp \rho_\epsilon) + \partial_\parallel(w_\epsilon \rho_\epsilon) = -\partial_\parallel(W_\epsilon \rho_\epsilon).$$

Proceeding similarly, we infer that ρ_ϵ converges strongly, up to a subsequence, to ρ in $\mathcal{C}([0,T],H^{s'})$ for all $s' < s$, that satisfies the equation:

$$\partial_t \rho + \nabla_\perp.(E^\perp \rho) + \partial_\parallel(w\rho) = 0.$$

One can likewise take limits in the Poisson equations. We finally obtain (1.8).

Step 4: Equations for the correctors

The final step relies on the following lemma proved in Grenier's paper [14, Proposition 3.3.4] (the main point is to notice that the application $\varphi \mapsto e^{\pm it/\sqrt{\epsilon}}\varphi$ is an isometry on $L^\infty(H^s)$ for any s).

Lemma 4.2. *There exist two correctors $E_+(t,x_\parallel)$ and $E_-(t,x_\parallel)$ in $\mathcal{C}([0,T],H^{s-1})$ such that, for all $s' < s$:*

- $\|\sqrt{\epsilon}E_\epsilon^{(1)} - e^{it/\sqrt{\epsilon}}E_+ - e^{-it/\sqrt{\epsilon}}E_-\|_{\mathcal{C}([0,T],H^{s'-1})} \to 0,$
- $\|W_\epsilon - \frac{1}{i}\left(e^{it/\sqrt{\epsilon}}E_+ - e^{-it/\sqrt{\epsilon}}E_-\right)\|_{\mathcal{C}([0,T],H^{s'-1})} \to 0.$

In particular we can deduce that:

$$e^{-it/\sqrt{\epsilon}}\sqrt{\epsilon}E_\epsilon^{(1)} \rightharpoonup E_+$$

(and similarly $e^{it/\sqrt{\epsilon}}\sqrt{\epsilon}E_\epsilon^{(1)} \rightharpoonup E_-$).

Then, the idea is to use Lemmas 4.1 and 4.2 and the wave equation (3.5) in order to obtain the equations satisfied by $E_\pm$. Let us show how one can obtain the equation for E_- (the method being similar for E_+). Let us denote $F_\epsilon = \sqrt{\epsilon}e^{it/\sqrt{\epsilon}}E_{\epsilon,\parallel}$. One can then observe that:

$$\epsilon\partial_t^2 E_{\epsilon,\parallel} + E_{\epsilon,\parallel} = e^{-it/\sqrt{\epsilon}}\left(\sqrt{\epsilon}\partial_t^2 F_\epsilon - 2i\partial_t F_\epsilon\right).$$

Furthermore, by Lemmas 4.1 and 4.2, F_ϵ weakly converges (in the distributional sense) to E_-. Using (3.5), we obtain an equation satisfied by F_ϵ:

$$\begin{aligned}\sqrt{\epsilon}\partial_t^2\partial_{x_\parallel}F_\epsilon - 2i\partial_t\partial_{x_\parallel}F_\epsilon &= e^{it/\sqrt{\epsilon}}\partial_{x_\parallel}^2\int \rho_\epsilon(w_\epsilon + W_\epsilon)^2 dx_\perp \\ &+ e^{it/\sqrt{\epsilon}}\partial_{x_\parallel}\int \rho_\epsilon(\epsilon\partial_{x_\parallel}\phi_\epsilon)dx_\perp - e^{it/\sqrt{\epsilon}}\epsilon\partial_{x_\parallel}[E_{\epsilon,\parallel}\partial_{x_\parallel}E_{\epsilon,\parallel}].\end{aligned} \tag{4.3}$$

We first show that $\sqrt{\epsilon}\partial_t^2\partial_{x_\parallel}F_{\epsilon,\parallel}$ weakly converges to 0 in the distributional sense. For this purpose let $\Psi(t,x_\parallel)$ a smooth test function compactly supported in $\mathbb{R}^{+*}\times\mathbb{R}$. We have by integration by parts:

$$\begin{aligned}\int \sqrt{\epsilon}\partial_t^2\partial_{x_\parallel}F_\epsilon\Psi dtdx_\parallel &= -\int \sqrt{\epsilon}\partial_t F_\epsilon\partial_t\partial_{x_\parallel}\Psi dtdx_\parallel \\ &= \int \sqrt{\epsilon}F_\epsilon\partial_t^2\partial_{x_\parallel}\Psi dtdx_\parallel,\end{aligned}$$

and we can conclude that the contribution of this three term vanishes as ϵ vanishes since F_ϵ is uniformly bounded in $\mathcal{C}([0,T],H^{s'-1}_{x_\parallel})$ by Lemma 4.1. Likewise, we show that $-2i\partial_t F_\epsilon$ converges in the distributional sense to $-2i\partial_t E_-$.

By Step 3, we recall that ρ_ϵ converges strongly (up to a subsequence) in $\mathcal{C}([0,T],H^{s'})$ (with $s'<s$). Let us show that $\epsilon\partial_{x_\parallel}\phi_\epsilon$ also converges strongly (up to a subsequence) in $\mathcal{C}([0,T],H^{s'})$. To that purpose, we rely once again on the Poisson equation satisfied by ϕ_ϵ, that we recall below:

$$-\epsilon^2\partial_{x_\parallel}^2\phi_\epsilon - \Delta_\perp\phi_\epsilon = \rho_\epsilon - \int\rho_\epsilon dx_\perp.$$

By the same symbolic analysis as before, one can easily check, using assumption (H), that $\epsilon\partial_{x_\parallel}\phi_\epsilon$ is uniformly bounded in $L^\infty_t(H^s_x)$. Deriving the Poisson equation with respect to time, we obtain:

$$-\epsilon^2\partial_{x_\parallel}^2\partial_t\phi_\epsilon - \Delta_\perp\partial_t\phi_\epsilon = \partial_t\rho_\epsilon - \int\partial_t\rho_\epsilon dx_\perp.$$

Using this time the uniform estimates on $\partial_t\rho_\epsilon$, we deduce that $\epsilon\partial_t\partial_{x_\parallel}\phi_\epsilon$ is uniformly bounded in $L^\infty_t(H^{s-2}_x)$.

Therefore, using the Aubin-Lions lemma, we have proved our claim.

We deduce that $\partial_{x_\parallel}\int\rho_\epsilon(\epsilon\partial_{x_\parallel}\phi_\epsilon)dx_\perp$ converge strongly (up to a subsequence) in $\mathcal{C}([0,T],H^{s'-1}_{x_\parallel})$, so we can see that:

$$e^{it/\sqrt{\epsilon}}\partial_{x_\parallel}\int\rho_\epsilon(\epsilon\partial_{x_\parallel}\phi_\epsilon)dx_\perp \rightharpoonup 0$$

in the sense of distributions.

In order to take the limit in the other terms, we have to be a little more precise. By Lemmas 4.1 and 4.2, we can write:

$$\begin{aligned}\sqrt{\epsilon}E_{\epsilon,\parallel} &= e^{it/\sqrt{\epsilon}}E_+ + e^{-it/\sqrt{\epsilon}}E_- + r_\epsilon,\\ W_\epsilon &= \frac{1}{i}\left(e^{it/\sqrt{\epsilon}}E_+ - e^{-it/\sqrt{\epsilon}}E_-\right) + s_\epsilon,\end{aligned}$$

where r_ϵ and s_ϵ converge strongly to 0 in $\mathcal{C}([0,T],H^{s'-1}_{x_\parallel})$. Consequently we deduce that $e^{it/\sqrt{\epsilon}}\epsilon\partial_{x_\parallel}[E_{\epsilon,\parallel}\partial_{x_\parallel}E_{\epsilon,\parallel}]$ converges to 0 in the sense of distributions. Indeed, we have:

$$\begin{aligned}e^{it/\sqrt{\epsilon}}\epsilon\partial_{x_\parallel}[E_{\epsilon,\parallel}\partial_{x_\parallel}E_{\epsilon,\parallel}] =& \frac{1}{2}e^{it/\sqrt{\epsilon}}\partial_{x_\parallel}^2\Big(r_\epsilon^2 + e^{2it/\sqrt{\epsilon}}E_+^2 + e^{-2it/\sqrt{\epsilon}}E_-^2 \\ &+ 2E_+E_- + 2e^{it/\sqrt{\epsilon}}E_+r_\epsilon + 2e^{-it/\sqrt{\epsilon}}E_-r_\epsilon\Big)\end{aligned}$$

Thus, as r_ϵ converges strongly to 0 in $\mathcal{C}([0,T],H^{s'-1}_{x_\parallel})$, there is no resonance effect and this converges to 0 in the sense of distributions. Now we write:

$$\partial_{x_\parallel}^2\int\rho_\epsilon(w_\epsilon+W_\epsilon)^2dx_\perp = \partial_{x_\parallel}^2\int\rho_\epsilon w_\epsilon^2dx_\perp + \partial_{x_\parallel}^2\left(\int\rho_\epsilon dx_\perp\right)W_\epsilon^2 + 2\partial_{x_\parallel}^2\int\rho_\epsilon w_\epsilon W_\epsilon dx_\perp.$$

Since $\partial_{x_\parallel}^2 \int \rho_\epsilon w_\epsilon^2 dx_\perp$ strongly converges in $\mathcal{C}([0,T],H^{s'-1}_{x_\parallel})$, the contribution of the first term, that is $e^{it/\sqrt{\epsilon}}\partial_{x_\parallel}^2 \int \rho_\epsilon w_\epsilon^2 dx_\perp$, vanishes. For the second term, we first notice that $\int \rho_\epsilon dx_\perp$ is strongly convergent in $\mathcal{C}([0,T],H^{s'}_{x_\parallel})$. Then, we can check as before that there is no resonance effect and the contribution of $e^{it/\sqrt{\epsilon}}\partial_{x_\parallel}^2 \left(\int \rho_\epsilon dx_\perp\right) W_\epsilon^2$ vanishes. For the last term, $\rho_\epsilon w_\epsilon$ strongly converges to ρv in $\mathcal{C}([0,T],H^{s'-1}_{x})$; using once again the decomposition of W_ϵ, we obtain that the limit in the distributional sense of $e^{it/\sqrt{\epsilon}}2\partial_{x_\parallel}^2 \int \rho_\epsilon w_\epsilon W_\epsilon dx_\perp$ is $2i\left(\int \rho v dx_\perp\right)\partial_{x_\parallel}(\partial_{x_\parallel}E_-)$.

As a result, $\partial_{x_\parallel}E_\pm$ satisfy the transport equations:

$$\partial_t(\partial_{x_\parallel}E_\pm) + \left(\int \rho v dx_\perp\right)\partial_{x_\parallel}(\partial_{x_\parallel}E_\pm) = 0.$$

There remains to provide some initial data for these equations. This is achieved thanks to the strong convergences in Lemma 4.2 that hold in particular for $t=0$. More precisely, we have by Lemma 4.2:

$$E_{+,|t=0} = \frac{1}{2}\lim_{\epsilon\to 0}[iW_{\epsilon,|t=0} + \sqrt{\epsilon}E_\epsilon^{(1)}], \quad E_{-,|t=0} = \frac{1}{2}\lim_{\epsilon\to 0}[-iW_{\epsilon,|t=0} + \sqrt{\epsilon}E_\epsilon^{(1)}].$$

By Lemma 4.1, (iii), we have:

$$\lim_{\epsilon\to 0} W_{\epsilon,|t=0} = \lim_{\epsilon\to 0}\int \rho_\epsilon v_\epsilon \, dx_\perp(0),$$

and by (iv) we have

$$\lim_{\epsilon\to 0}\sqrt{\epsilon}E_\epsilon^{(1)} = \lim_{\epsilon\to 0} -\sqrt{\epsilon}\partial_{x_\parallel}V_\epsilon(0).$$

This yields the initial conditions (2.6) and (2.7).

The proof of the theorem is now complete.

5 Discussion on the sharpness of the results

5.1 On the analytic regularity

Let us recall that the multi-fluid system (2.2) is ill-posed in Sobolev spaces (see [4]), because of the two-stream instabilities (remind that this is due to the coupling between the different phases of the fluid).

For system (1.8), we expect the situation to be similar. Due to the dependence on $x_\perp$ and the constraint $\int \rho dx_\perp = 1$, system (1.8) is by nature a coupled multi-fluid system. Nevertheless, one could maybe imagine that the dynamics in the $x_\perp$ variable could yield some mixing in $x_\perp$ and $x_\parallel$ (in the spirit of hypoellipticity results) and thus could perhaps bring stability. Here we explain why this is not the case.

The idea is to consider for (1.8) shear flows like initial data. This will allow to exactly recover the multi-fluid equations (2.2). Writing $x_\perp = (x_1, x_2)$, we take:

$$E_0^\perp = (0, \varphi(x_1, x_\parallel), 0),$$

and consequently since by definition:

$$\rho_0 = \mathrm{div}_x\, E_0 + 1,$$

we infer that $\rho_0 = \nabla_\perp \wedge E_0^\perp = -\varphi'(x_1, x_\|) + 1$. We also assume that $v_0(x_1, x_\|)$ does not depend on x_2.

Then we observe that:

$$\nabla_\perp.(E_0^\perp \rho_0) = 0,$$
$$\nabla_\perp.(E_0^\perp v_0) = 0.$$

With such initial data, system (1.8) reduces to:

$$\begin{cases} \partial_t \rho + \partial_\|(v_\| \rho) = 0 \\ \partial_t v_\| + v_\| \partial_\|(v_\|) = -\partial_\| p(t, x_\|) \\ \int \rho dx_1 = 1, \end{cases} \tag{5.1}$$

and we observe that there is no more dynamics in the $x_\perp$ variable. This is nothing but system (2.2) in dimension 1, with $M = [0, 1[$ and μ the Lebesgue measure.

Now, let us consider measure type of data in the x_1 variable for ρ and v (this corresponds to a “degenerate” version of the shear flows defined above). In particular if we choose:

$$\varphi = \frac{1}{2}\delta_{x_1 \leq \frac{1}{4}} \rho_{0,1}(x_\|) + \frac{1}{2}\delta_{x_1 \leq \frac{1}{2}} \rho_{0,2}(x_\|),$$

we get:

$$\begin{aligned} \rho_0 &= \frac{1}{2}\delta_{x_1 = \frac{1}{4}} \rho_{0,1}(x_\|) + \frac{1}{2}\delta_{x_2 = \frac{1}{2}} \rho_{0,2}(x_\|), \\ v_0 &= \frac{1}{2}\delta_{x_1 = \frac{1}{4}} v_{0,1}(x_\|) + \frac{1}{2}\delta_{x_1 = \frac{1}{2}} v_{0,2}(x_\|) \end{aligned} \tag{5.2}$$

and we obtain the following system for $\alpha = 1, 2$:

$$\begin{cases} \partial_t \rho_\alpha + \partial_\|(v_\alpha \rho_\alpha) = 0 \\ \partial_t v_\alpha + v_\alpha \partial_\|(v_\alpha) = -\partial_\| p(t, x_\|) \\ \rho_1 + \rho_2 = 1. \end{cases} \tag{5.3}$$

This particular system was given as an example by Brenier in [4] to illustrate ill-posedness in Sobolev spaces of the multi-fluid equations. Indeed let us first denote $q = \rho_1 v_1$. Using the constraint $\rho_1 + \rho_2 = 1$, we easily obtain that

$$p_\| = -q^2 \left(\frac{1}{\rho_1} + \frac{1}{1 - \rho_1} \right).$$

We can then observe that the system:

$$\begin{cases} \partial_t \rho_1 + \partial_\| q = 0 \\ \partial_t q + \partial_\|(\frac{q^2}{\rho_1}) = -\rho_1 \partial_\| p(t, x_\|) \end{cases} \tag{5.4}$$

is elliptic in space-time, and consequently it is ill-posed in Sobolev spaces.

Actually this example is not completely satisfying, since it is singular in x_1. Nevertheless we can consider the convolution of this initial data with a standard mollifier, which yields the same qualitative behaviour.

5.2 On the analytic regularity in the perpendicular variable

We observe that if the initial datum $(\rho(0), v(0))$ does not depend on $x_\parallel$, then the fluid system (1.8) reduces to:

$$\begin{cases} \partial_t \rho + \nabla_\perp.(E^\perp \rho) = 0 \\ \partial_t v_\parallel + \nabla_\perp.(E^\perp v_\parallel) = 0 \\ E^\perp = \nabla^\perp \Delta_\perp^{-1} \left(\rho - \int \rho dx_\perp\right) \\ \int \rho dx_\perp = 1. \end{cases} \tag{5.5}$$

Thus, ρ satisfies $2D$ incompressible Euler system, written in vorticity formulation. This systems admits a unique global strong solution provided that $\rho(0) \in H^s(\mathbb{T}^2)$ (with $s > 1$), by a classical result of Kato [21] and even a unique global weak solution provided that $\rho(0) \in L^\infty(\mathbb{T}^2)$, by a classical result of Yudovic [29].

In the other hand, $v_\parallel$ satisfied a transport equation with the force field $E^\perp$. If we only assume for instance that v_0 is a positive Radon measure, then using the classical log-Lipschitz estimate on $E^\perp$ (we refer to [23, Chapter 8]), we get a unique global weak solution $v_\parallel$ by the method of characteristics.

One could think that it should be possible to build solutions to the final fluid system (1.8) with similar "weak" regularity in the $x_\perp$ variable (while keeping analyticity in the $x_\parallel$ variable). Actually this is not possible in general: this is related to the fact that $E^\perp$ depends also on $x_\parallel$ in general and this entails that we also need analytic regularity in the $x_\perp$ variable to get analytic regularity in the $x_\parallel$ variable (see estimations such as (3.26)).

5.3 On the local in time existence

In [5], Brenier considers potential velocity fields, that are velocity fields of the form $v_\Theta = \nabla_x \Phi_\Theta$, for the multi-fluid system:

$$\begin{cases} \Theta = 1, ..., M \quad M \in \mathbb{N}^* \\ \partial_t \rho_\Theta + \mathrm{div}(\rho_\Theta v_\Theta) = 0 \\ \partial_t v_\Theta + v_\Theta.\nabla(v_\Theta) = -\nabla_x p \\ \sum_{\Theta=1}^M \rho_\Theta = 1. \end{cases} \tag{5.6}$$

In this case the equation on the velocities becomes:

$$\partial_t \Phi_\Theta + \frac{1}{2}|\nabla_x \Phi_\Theta|^2 + p = 0. \tag{5.7}$$

It is proved in [5] that any strong solution satisfying

$$\inf_{\Theta,t,x} \rho_\Theta(t,x) > 0$$

can not be global in time unless the initial energy vanishes:

$$\sum_{\Theta=1}^M \int \rho_{\Theta,t=0} |u_{\Theta,t=0}|^2 dx = 0. \tag{5.8}$$

This striking result relies on a variational interpretation of these Euler equations. Using the same particular initial data as in section 5.1, this indicates that for system (1.8) also, there is no global strong solution, unless there is no dependence on $x_\perp$ or $x_\parallel$.

Indeed, we observe that if the initial datum $(\rho(0), v(0))$ does not depend on $x_\perp$, the fluid system (1.8) does not make sense anymore (as for incompressible Euler in dimension 1). When the initial datum $(\rho(0), v(0))$ does not depend on $x_\parallel$, we have seen that we recover $2D$ incompressible Euler and there is indeed global existence (of strong or weak solutions).

5.4 The relative entropy method applied to a toy model : failure of the multi-current limit

5.4.1 The toy model

It seems very appealing to try to use the relative entropy method (which was introduced by Brenier [4] for Vlasov type of systems) to study the limit $\epsilon \to 0$, as it would open the way to the study of the limit for solutions to the initial system (1.1) with low regularity. The only requirements would be that the initial data of (1.1) is closed in some sense (which will be made precise later) to a Dirac mass in velocity, and that the two first moments of the initial data are in a small neighborhood (say in L^2 topology) of the smooth initial data for the limit system (1.8). Nevertheless it is not possible to overcome the two-stream instabilities in this framework. We intend here to show why.

The toy model we consider in this paragraph is the following:

$$\begin{cases} \partial_t f^\theta_\epsilon + v.\nabla_x f^\theta_\epsilon + E_\epsilon.\nabla_v f^\theta_\epsilon = 0 \\ E_\epsilon = -\nabla_x V_\epsilon \\ -\epsilon\Delta_x V_\epsilon = \int\int f^\theta_\epsilon dv d\mu - 1 \\ f^\theta_{\epsilon,t=0} = f^\theta_{\epsilon,0}, \quad \int\int f^\theta_\epsilon dv dx d\theta = 1. \end{cases} \tag{5.9}$$

with $t > 0$, $x \in \mathbb{T}^3$, $v \in \mathbb{R}^3$ and where θ lies in $[0,1]$ equipped with a probability measure μ which is:

- either a sum of Dirac masses with total mass 1, such as:

  $$\mu = \sum_{i=0}^{N-1} \frac{1}{N}\delta_{\theta=i/N}.$$

  In this case, we model a plasma made of N phases (or N types of charged particles).

- or the Lebesgue measure, in which case we model a continuum of phases.

Actually, we could have considered more general probability measures but we restrict to these cases for simplicity. This system can be seen as the kinetic counterpart of a simplified version of (1.7), which focuses on the unstable feature of the system. Of course we could have considered directly the fluid version, that is:

$$\begin{cases} \partial_t \rho^\theta_\epsilon + \nabla_x.(\rho^\theta_\epsilon u^\theta_\epsilon) = 0 \\ \partial_t u^\theta_\epsilon + u^\theta_\epsilon.\nabla_x u^\theta_\epsilon = E_\epsilon \\ E_\epsilon = -\nabla_x V_\epsilon \\ -\epsilon\Delta_x V_\epsilon = \int \rho^\theta_\epsilon d\mu - 1 \end{cases} \tag{5.10}$$

but the proofs are essentially the same and the study of system (5.9) has some interests of its own.

One can observe that the energy associated to (5.9) is the following non-increasing (formally conserved) functional:

$$\mathcal{E}_\epsilon(t) = \frac{1}{2}\int\int f^\theta_\epsilon |v|^2 dv dx d\mu + \frac{1}{2}\epsilon\int |\nabla_x V_\epsilon|^2 dx. \tag{5.11}$$

We assume that there exists a constant $K > 0$ independent of ϵ, such as $\mathcal{E}_\epsilon(0) \leq K$. We also assume that $f^\theta_0 \in L^\infty_\theta L^1_{x,v} \cap L^\infty_\theta L^\infty_{x,v}$, uniformly in ϵ. Then we can consider global weak solutions $(f^\theta_\epsilon, V_\epsilon)$ to (5.9), in the sense of Arsenev [1].That these solutions exist follows from a slight adaptation of the original proof in [1], which dealt with the usual Vlasov-Poison equation. These solutions satisfy that uniformly in ϵ, $f^\theta_\epsilon \in L^\infty_{t,\theta} L^1_{x,v} \cap L^\infty_{t,\theta} L^\infty_{x,v}$. In addition, for any ϵ and any $t \geq 0$:

$$\mathcal{E}_\epsilon(t) \leq K. \tag{5.12}$$

Let (ρ^θ, u^θ) be the local strong solution, defined on $[0,T]$, to the system:

$$\begin{cases} \partial_t \rho^\theta + \nabla_x.(\rho^\theta u^\theta) = 0 \\ \partial_t u^\theta + u^\theta.\nabla_x u^\theta = -\nabla_x V \\ \int \rho^\theta d\mu = 1. \end{cases} \tag{5.13}$$

with inital data $(\rho^\theta_0, u^\theta_0)$ (which we actually have to take with analytic regularity in general). Observe here that the "incompressibility in average" constraint reads:

$$\nabla_x. \int \rho^\theta u^\theta d\mu = 0. \tag{5.14}$$

The case where u^θ_0 genuinely depends on θ corresponds to the setting for two-stream instabilities [8]. In this case, as expected, we will not be able to conclude. On the contrary, when u^θ_0 does not depend on θ, this precisely corresponds to the case where two-stream instabilities are avoided, and in that particular case, the relative entropy method will yield convergence: this is the result of Proposition 5.1.

5.4.2 The relative entropy method

Following the approach of Brenier [4] for the quasineutral limit of the Vlasov-Poisson equation with a single phase, we consider the relative entropy (built as a modulation of the energy $\mathcal{E}_\epsilon$):

$$\mathcal{H}_\epsilon(t) = \frac{1}{2}\int\int f^\theta_\epsilon |v - u^\theta(t,x)|^2 dvdxd\mu + \frac{1}{2}\epsilon \int |\nabla_x V_\epsilon - \nabla_x V|^2 dx. \tag{5.15}$$

We assume that the system is well prepared in the sense that $\mathcal{H}_\epsilon(0) \to 0$ when $\epsilon \to 0$. The goal is to find some stability inequality in order to show that we also have $\mathcal{H}_\epsilon(t) \to 0$ for $t \in [0,T]$.

We have, since the energy is non-increasing:

$$\begin{aligned} \frac{d}{dt}\mathcal{H}_\epsilon(t) \leq & \int\int \partial_t f^\theta_\epsilon \left(\frac{1}{2}|u^\theta|^2 - v.u^\theta\right) dvdxd\mu + \int\int f^\theta_\epsilon \partial_t \left(\frac{1}{2}|u^\theta|^2 - v.u^\theta\right) dvdxd\mu \\ & + \frac{1}{2}\epsilon \int \partial_t |\nabla_x V|^2 dx - \epsilon \int \nabla_x V_\epsilon . \partial_t \nabla_x V dx - \epsilon \int \partial_t \nabla_x V_\epsilon . \nabla_x V dx. \end{aligned} \tag{5.16}$$

We clearly have $\epsilon \int \partial_t |\nabla_x V|^2 dx = \mathcal{O}(\epsilon)$. Moreover, we get, by Cauchy-Schwarz inequality:

$$\epsilon \Big| \int \nabla_x V_\epsilon . \partial_t \nabla_x V dx \Big| \leq \sqrt{\epsilon} \|\sqrt{\epsilon}\nabla_x V_\epsilon\|_{L^\infty_t L^2_x} \|\partial_t \nabla_x V\|_{L^\infty_t L^2_x},$$

which is of order $\mathcal{O}(\sqrt{\epsilon})$ by the conservation of energy.

For the last term of (5.16), we compute with successive integrations by parts:

$$
\begin{aligned}
-\epsilon \int \partial_t \nabla_x V_\epsilon . \nabla_x V dx &= \epsilon \int \partial_t \Delta_x V_\epsilon V dx \\
&= - \int \partial_t \left(\int f_\epsilon^\theta dv d\mu \right) V dx \\
&= \int \nabla_x . \left(\int f_\epsilon^\theta v \, dv d\mu \right) V dx \\
&= - \int \left(\int f_\epsilon^\theta v \, dv d\mu \right) . \nabla_x V dx.
\end{aligned} \tag{5.17}
$$

In this computation we have used the Poisson equation as well as the local conservation of mass (obtained by integrating the Vlasov equation in (5.9) against v):

$$
\partial_t \int f_\epsilon^\theta dv + \nabla_x . \left(\int v f_\epsilon^\theta dv \right) = 0.
$$

In the other hand we can compute:

$$
\begin{aligned}
&\int\int \partial_t f_\epsilon^\theta \left(\frac{1}{2} |u^\theta|^2 - v.u^\theta \right) dv dx d\mu + \int\int f_\epsilon^\theta \partial_t \left(\frac{1}{2} |u^\theta|^2 - v.u^\theta \right) dv dx d\mu \\
&= - \int\int (v.\nabla_x f_\epsilon^\theta + E_\epsilon . \nabla_v f_\epsilon^\theta) \left(\frac{1}{2} |u^\theta|^2 - v.u^\theta \right) dv dx d\mu + \int\int f_\epsilon^\theta (u^\theta - v) . \partial_t u^\theta dv dx d\mu \\
&= - \int\int f_\epsilon^\theta v.((u^\theta - v).\nabla_x u^\theta) dv dx d\mu - \int f_\epsilon^\theta E_\epsilon . u^\theta dv dx d\mu + \int\int f_\epsilon^\theta (u^\theta - v). \partial_t u^\theta dv dx d\mu \\
&= \int\int f_\epsilon^\theta (u^\theta - v).((u^\theta - v).\nabla_x u \theta) dv dx d\mu + \int\int f_\epsilon^\theta (u^\theta - v).(\partial_t u^\theta + u^\theta . \nabla_x u^\theta) dv dx d\mu \\
&- \int f_\epsilon^\theta E_\epsilon . u^\theta dv dx d\mu.
\end{aligned} \tag{5.18}
$$

All the trouble comes from the last term:

$$
\int f_\epsilon^\theta E_\epsilon . u^\theta dv dx d\mu.
$$

When no assumption is made on u^θ, it can be of order $\mathcal{O}(1/\sqrt{\epsilon})$. This wild term can be interpreted as the appearance of the two-stream instabilities. Therefore we have to make an additional assumption in order to avoid this instability. This is done by assuming that u^θ initially does not depend on θ (which yields that u^θ does not depend on θ by uniqueness), in which case we can write:

$$
u^\theta = u
$$

and consequently, we have

$$
- \int f_\epsilon^\theta E_\epsilon . u \, dv dx d\mu = \int (\epsilon \Delta_x V_\epsilon - 1) E_\epsilon . u dx. \tag{5.19}
$$

We first compute:

$$
\begin{aligned}
- \int \epsilon \int \Delta_x V_\epsilon \nabla_x V_\epsilon . u dx &= -\epsilon \int \nabla_x : (\nabla_x V_\epsilon \otimes \nabla_x V_\epsilon) u dx + \epsilon \int \frac{1}{2} \nabla_x |\nabla_x V_\epsilon|^2 u dx \\
&= \epsilon \int D(u) : (\nabla_x V_\epsilon \otimes \nabla_x V_\epsilon) dx - \epsilon \int \frac{1}{2} |\nabla_x V_\epsilon|^2 \operatorname{div}_x u dx,
\end{aligned}
$$

with $D(u) = \frac{1}{2}\left(\partial_{x_i} u_j + \partial_{x_j} u_i\right)_{i,j}$.

In addition, the incompressibility constraint (5.14) becomes $\nabla_x . u = 0$, and thus:

$$\int E_\epsilon . u dx = \int V_\epsilon \nabla_x . u dx = 0.$$

Gathering all pieces together, we obtain:

$$\begin{aligned} \mathcal{H}_\epsilon(t) \leq \mathcal{H}_\epsilon(0) + R_\epsilon(t) + C \int_0^t \|\nabla_x u\| \mathcal{H}_\epsilon(s) ds \\ + \int_0^t \int \int f_\epsilon^\theta (u - v)(\partial_t u + u . \nabla_x u) d\mu dv dx ds - \int_0^t \int \int f_\epsilon^\theta v . \nabla_x V d\mu dv dx ds, \end{aligned} \tag{5.20}$$

where $C > 0$ is a universal constant, $R_\epsilon(t) \to 0$ as ϵ goes to 0. Furthermore, we remark that:

$$\int \left(\int f_\epsilon^\theta dv d\mu \right) u . \nabla_x V dv = \int u . \nabla_x V - \epsilon \int \Delta_x V_\epsilon u . \nabla_x V \tag{5.21}$$

The first term is equal to 0 according to the incompressibility constraint, while the second is of order $\mathcal{O}(\sqrt{\epsilon})$, by the energy inequality. We finally get the stability inequality:

$$\begin{aligned} \mathcal{H}_\epsilon(t) \leq \mathcal{H}_\epsilon(0) + \tilde{R}_\epsilon(t) + C \int_0^t \|\nabla_x u\| \mathcal{H}_\epsilon(s) ds \\ + \int_0^t \int \int f_\epsilon^\theta (u - v)(\partial_t u + u . \nabla_x u + \nabla_x V) d\mu dv dx ds, \end{aligned} \tag{5.22}$$

where $C > 0$ is a universal constant, $\tilde{R}_\epsilon(t) \to 0$ as ϵ goes to 0 and the last term is 0 by definition of (u, V).

As as result, by Gronwall's inequality, we infer that $\mathcal{H}_\epsilon(t) \to 0$, uniformly locally in time. To conclude, by a classical interpolation argument using the fact that $f_\epsilon |v|^2$ is uniformly in $L^\infty_t L^1_{x,v,\theta}$ and that f_ϵ is uniformly in $L^\infty_t L^1_{t,x,v}$, we infer that $\rho_\epsilon^\theta := \int f_\epsilon^\theta dv$ and $J_\epsilon^\theta := \int f_\epsilon^\theta v dv$ are uniformly bounded in $L^\infty_t(L^1_{\theta,x})$. Thus, up to a subsequence, there exist ρ^θ and J^θ (at least in $L^\infty_t(L^1_{\theta,x})$) such that ρ_ϵ^θ weakly converges in the sense of measures to ρ^θ (resp. J_ϵ^θ to J^θ). Passing to the limit in the local conservation of charge, which reads:

$$\partial_t \rho_\epsilon^\theta + \nabla_x . J_\epsilon^\theta = 0,$$

we obtain:

$$\partial_t \rho^\theta + \nabla_x . J^\theta = 0.$$

The goal is now to prove that $J^\theta = \rho^\theta u$.

By a simple use of Cauchy-Schwarz inequality, we have:

$$\int \int \frac{|\rho_\epsilon^\theta u - J_\epsilon^\theta|^2}{\rho_\epsilon^\theta} dx d\mu \leq \int \int f_\epsilon^\theta |v - u|^2 dv dx d\mu. \tag{5.23}$$

Using a classical convexity argument due to Brenier [6], one can prove that the functional $(\rho, J) \mapsto \int \frac{|\rho u - J|^2}{\rho} dx d\mu$ is lower semi-continuous with respect to the weak convergence of measures. We finally obtain by passing to the limit that:

$$J^\theta = \rho^\theta u.$$

By uniqueness of the solution to the limit system, provided that the whole sequence $(\rho_{\epsilon,0}^\theta)$ weakly converges to ρ_0^θ, we obtain the convergences without having to extract subsequences.

Finally we have proved the result:

Proposition 5.1. *Let $(f_\epsilon^\theta, V_\epsilon)$ be a global weak solution in the sense of Arsenev to (5.9). Assume that for some functions (ρ_0^θ, u_0) in $(L^1_{\theta,x} \times H^s_x)$, with $s > 5/2$, (we emphasize on the fact that u_0 does not depend on θ, in order to avoid two-stream instabilities) satisfying*

$$\begin{cases} \int \rho_0^\theta d\mu = 1, \\ \nabla_x . u_0 = 0, \end{cases} \tag{5.24}$$

and such that we initially have:

$$\frac{1}{2}\int\int f^\theta_{\epsilon,t=0}|v - u_0(x)|^2 dvdxd\mu + \frac{1}{2}\epsilon\int |\nabla_x V_{\epsilon,t=0} - \nabla_x V_{t=0}|^2 dx \to 0 \tag{5.25}$$

and $\int f_\epsilon^\theta dv \rightharpoonup \rho_0^\theta$ in the weak L^1 sense.

Let (u, V) is the (unique) local strong solution (defined on $[0, T[$) to the incompressible Euler system:

$$\begin{cases} \partial_t u + u.\nabla_x u = -\nabla_x V \\ \nabla_x . u = 0, \end{cases} \tag{5.26}$$

with initial data $u(t = 0) = u_0$. Then for all $t \in [0, T[$,

$$\frac{1}{2}\int\int f_\epsilon^\theta |v - u(t, x)|^2 dvdxd\mu + \frac{1}{2}\epsilon\int |\nabla_x V_\epsilon - \nabla_x V|^2 dx \to 0, \tag{5.27}$$

where (u, V) is the local strong solution to the incompressible Euler system:

$$\begin{cases} \partial_t u + u.\nabla_x u = -\nabla_x V \\ \nabla_x . u = 0. \end{cases} \tag{5.28}$$

Moreover, $\rho_\epsilon^\theta := \int f_\epsilon^\theta dv$ converges in the weak L^1 sense to ρ^θ the unique solution to:

$$\partial_t \rho^\theta + u.\nabla_x \rho^\theta = 0, \tag{5.29}$$

with $\rho^\theta(t = 0) = \rho_0^\theta$ and $J_\epsilon^\theta := \int f_\epsilon^\theta v dv$ converges in the weak L^1 sense to $\rho^\theta u$.

6 Conclusion

In this work, we have provided a first analysis of the mathematical properties of the three-dimensional finite Larmor radius approximation (FLR), for electrons in a fixed background of ions. We have shown that the limit is illposed in the sense that we have to restrict to data with both particular profiles and analytic regularity. In particular, we have pointed out that the analytic assumption is not only a mere technical assumption, but is necessary if one choses to consider strong solutions. In addition, the results are only local-in-time.

On the other hand, we proved in [18] that the FLR approximation for ions with massless electrons is by opposition very stable, in the sense that we can deal with initial data with no prescribed profile and weak (that is in a Lebesgue space) regularity.

This rigorously justifies why physicists rather consider the equations on ions rather than those on electrons, especially for numerical experiments (we refer for instance to Grandgirard et al. [13]).

7 Appendix : Formal derivation of the drift-fluid problem

Scaling of the Vlasov equation

Let us recall that our purpose is to describe the behaviour of a gas of electrons in a neutralizing background of ions at thermodynamic equilibrium, submitted to a large magnetic field. For simplicity, we consider a magnetic field with a fixed direction $e_\|$ (also denoted by e_z) and a fixed large magnitude $\bar{B}$.

Because of the strong magnetic field, the dynamics of particles in the parallel direction $e_\|$ is completely different to their dynamics in the orthogonal plane. We therefore consider anisotropic characteristic spatial lengths in order to consider dimensionless quantities:

$$\tilde{x}_\perp = \frac{x_\perp}{L_\perp}, \quad \tilde{x}_\perp = \frac{x_\|}{L_\|},$$

$$\tilde{t} = \frac{t}{\tau}, \quad \tilde{v} = \frac{v}{v_{th}},$$

$$f(t, x_\perp, x_\|, v) = \bar{f}\tilde{f}(\tilde{t}, \tilde{x}_\perp, \tilde{x}_\|, \tilde{v}) \quad V(t, x_\perp, x_\|) = \bar{V}\tilde{V}(\tilde{t}, \tilde{x}_\perp, \tilde{x}_\|) \quad E(t, x_\perp, x_\|) = \bar{E}\tilde{E}(\tilde{t}, \tilde{x}_\perp, \tilde{x}_\|).$$

This yields:

$$\left\{ \begin{array}{l} \partial_{\tilde{t}}\tilde{f}_\epsilon + \frac{v_{th}\tau}{L_\perp}\tilde{v}_\perp.\nabla_{\tilde{x}_\perp}\tilde{f}_\epsilon + \frac{v_{th}\tau}{L_\|}\tilde{v}_\|.\nabla_{\tilde{x}_\|}\tilde{f}_\epsilon + \left(\frac{e\bar{E}\tau}{mv_{th}}\tilde{E}_\epsilon + \frac{e\bar{B}}{m}\tau\tilde{v}\wedge e_\|\right).\nabla_{\tilde{v}}\tilde{f}_\epsilon = 0 \\ \frac{\bar{E}}{\bar{V}}\tilde{E}_\epsilon = \left(-\frac{1}{L_\perp}\nabla_{\tilde{x}_\perp}\tilde{V}_\epsilon, -\frac{1}{L_\|}\nabla_{\tilde{x}_\|}\tilde{V}_\epsilon\right) \\ -\frac{\epsilon_0\bar{V}}{L_\perp^2}\Delta_{\tilde{x}_\perp}\tilde{V}_\epsilon - \frac{\epsilon_0\bar{V}}{L_\|^2}\Delta_{\tilde{x}_\|}\tilde{V}_\epsilon = e\bar{f}v_{th}^3\left(\int \tilde{f}_\epsilon d\tilde{v} - 1\right) \\ \tilde{f}_{\epsilon,|\tilde{t}=0} = \tilde{f}_{0,\epsilon}, \quad \bar{f}L_\perp^2 L_\| v_{th}^3 \int \tilde{f}_{0,\epsilon}d\tilde{v}d\tilde{x} = 1. \end{array} \right. \tag{7.1}$$

In order to keep normalization, it is first natural to set $\bar{f}L_\perp^2 L_\| v_{th}^3 = 1$.

We set now $\Omega = \frac{e\bar{B}}{m}$: this is the cyclotron frequency (also referred to as the gyrofrequency). We also consider the so-called electron Larmor radius (or electron gyroradius) r_L defined by:

$$r_L = \frac{v_{th}}{\Omega} = \frac{mv_{th}}{e\bar{B}} \tag{7.2}$$

This quantity can be physically understood as the typical radius of the helix around axis $e_\|$ described by the particles, due to the intense magnetic field.

We also introduce the so-called Debye length:

$$\lambda_D^2 = \frac{\epsilon_0\bar{V}}{e\bar{f}v_{th}^3},$$

which is interpreted as the typical length above which the plasma can be interpreted as being neutral.

The Vlasov equation now reads:

$$\partial_{\tilde{t}}\tilde{f}_\epsilon + \frac{r_L}{L_\perp}\Omega\tau\tilde{v}_\perp.\nabla_{\tilde{x}_\perp}\tilde{f}_\epsilon + \frac{r_L}{L_\|}\Omega\tau\tilde{v}_\|.\nabla_{\tilde{x}_\|}\tilde{f}_\epsilon + \left(\frac{\bar{E}}{\bar{B}v_{th}}\Omega\tau\tilde{E}_\epsilon + \Omega\tau\tilde{v}\wedge e_\|\right).\nabla_{\tilde{v}}\tilde{f}_\epsilon = 0.$$

The strong magnetic field ordering consists in:

$$\Omega\tau = \frac{1}{\epsilon}, \quad \frac{\bar{E}}{\bar{B}v_{th}} = \epsilon,$$

with $\epsilon > 0$ is a small parameter.

The spatial scaling we perform is the so-called finite Larmor radius scaling (see Frénod and Sonnendrucker [10] for a reference in the mathematical literature): basically the idea is to consider the typical perpendicular spatial length $L_\perp$ with the same order as the so-called electron Larmor radius. This allows to describe the turbulent behaviour of the plasma at fine scales, see [22]. On the contrary, the parallel observation length $L_\parallel$ is taken much larger:

$$\frac{r_L}{L_\perp} = 1, \quad \frac{r_L}{L_\parallel} = \epsilon. \tag{7.3}$$

This is typically an anisotropic situation.

This particular scaling allows, at least in a formal sense, to observe more precise effects in the orthogonal plane than with the isotropic scaling (studied for instance in [12]):

$$\frac{r_L}{L_\perp} = \epsilon, \quad \frac{r_L}{L_\parallel} = \epsilon.$$

In particular we wish to observe the so-called electric drift $E^\perp$ (also referred to as the $E \times B$ drift) whose effect is of great concern in tokamak physics (see [17] for instance).

The quasineutral ordering we adopt is the following:

$$\frac{\lambda_D}{L_\parallel} = \sqrt{\epsilon}. \tag{7.4}$$

After straightforward calculations (we refer to [10] for details), we get the following Vlasov-Poisson system in dimensionless form, for $t \geq 0, x = (x_\perp, x_\parallel) \in \mathbb{T}^2 \times \mathbb{T}, v = (v_\perp, v_\parallel) \in \mathbb{R}^2 \times \mathbb{R}$:

$$\begin{cases} \partial_t f_\epsilon + \frac{v_\perp}{\epsilon}.\nabla_x f_\epsilon + v_\parallel.\nabla_x f_\epsilon + (E_\epsilon + \frac{v\wedge e_z}{\epsilon}).\nabla_v f_\epsilon = 0 \\ E_\epsilon = (-\frac{1}{\epsilon}\nabla_{x_\perp} V_\epsilon, -\nabla_{x_\parallel} V_\epsilon) \\ -\epsilon\Delta_{x_\parallel} V_\epsilon - \frac{1}{\epsilon}\Delta_{x_\perp} V_\epsilon = \int f_\epsilon dv - \int f_\epsilon dv dx \\ f_{\epsilon,t=0} = f_{\epsilon,0}. \end{cases} \tag{7.5}$$

which yields, after setting $\overline{V}_\epsilon = \frac{1}{\epsilon} V_\epsilon$ (by a slight abuse of notation, we still denote V_ϵ instead of $\overline{V}_\epsilon$),

$$\begin{cases} \partial_t f_\epsilon + \frac{v_\perp}{\epsilon}.\nabla_x f_\epsilon + v_\parallel.\nabla_x f_\epsilon + (E_\epsilon + \frac{v\wedge e_z}{\epsilon}).\nabla_v f_\epsilon = 0 \\ E_\epsilon = (-\nabla_{x_\perp} V_\epsilon, -\epsilon\nabla_{x_\parallel} V_\epsilon) \\ -\epsilon^2\Delta_{x_\parallel} V_\epsilon - \Delta_{x_\perp} V_\epsilon = \int f_\epsilon dv - \int f_\epsilon dv dx \\ f_{\epsilon,t=0} = f_{\epsilon,0}. \end{cases} \tag{7.6}$$

Remark 7.1. *It seems physically relevant to consider scalings such as:*

$$\lambda_D / L_\parallel \sim \epsilon^\alpha, \tag{7.7}$$

with $\alpha \geq 1$. However with such a scaling, the systems seem too degenerate with respect to ϵ and we have not been able to handle this situation. The scaling we study is nevertheless relevant for some extreme magnetic regimes in tokamaks .

Hydrodynamic equations

In order to isolate this quasineutral problem, thanks to the linearity of the Poisson equation, we split the electric field into two parts:

$$
\begin{cases}
E_\epsilon = E^1_\epsilon + E^2_\epsilon,\\
E^1_\epsilon = (-\nabla_{x_\perp} V^1_\epsilon, -\epsilon\nabla_{x_\|} V^1_\epsilon),\\
-\epsilon^2\Delta_{x_\|} V^1_\epsilon - \Delta_{x_\perp} V^1_\epsilon = \int f_\epsilon dv - \int f_\epsilon dv dx_\perp,\\
E^2_\epsilon = -\partial_{x_\|} V^2_\epsilon,\\
-\epsilon\Delta_{x_\|} V^2_\epsilon = \int f_\epsilon dv dx_\perp - \int f_\epsilon dv dx.
\end{cases}
\tag{7.8}
$$

In order to make the fast oscillations in time due to the singularly penalized operator $\frac{v_\perp}{\epsilon}.\nabla_x$ disappear, we perform the same change of variables as in [11], to get the so-called gyro-coordinates:

$$
x_g = x_\perp + v^\perp, v_g = v_\perp. \tag{7.9}
$$

We easily compute the equation satisfied by the new distribution function $g_\epsilon(t, x_g, v_g, v_\|) = f_\epsilon(t, x, v)$.

$$
\begin{aligned}
&\partial_t g_\epsilon + v_\| \partial_{x_\|} g_\epsilon + E^1_{\epsilon,\|}(t, x_g - v_g^\perp)\partial_{v_\|} g_\epsilon + E^2_\epsilon(t, x_{g,\|})\partial_{v_\|} g_\epsilon\\
&+E^1_{\epsilon,\perp}(t, x_g - v_g^\perp).(\nabla_{v_g} g_\epsilon - \nabla^\perp_{x_g} g_\epsilon) + \frac{1}{\epsilon} v_g^\perp.\nabla_{v_g} g_\epsilon = 0.
\end{aligned}
$$

Notice here that in the process, the so-called electric drift $E^\perp$ appears since:

$$
-E^1_{\epsilon,\perp}(t, x_g - v_g^\perp).\nabla^\perp_{x_g} g_\epsilon = E^{1,\perp}_\epsilon(t, x_g - v_g^\perp).\nabla_{x_g} g_\epsilon.
$$

The equation satisfied by the charge density $\rho_\epsilon = \int g_\epsilon dv$ states:

$$
\partial_t \rho_\epsilon + \partial_{x_\|} \int v_\| g_\epsilon dv + \nabla^\perp_{x_g}. \int E^1_{\epsilon,\perp}(t, x_g - v_g^\perp) g_\epsilon dv = 0, \tag{7.10}
$$

One can observe that since $E^1_{\epsilon,\perp}$ is a gradient:

$$
\mathrm{div}_{v_g}\, E^1_{\epsilon,\perp}(t, x_g - v_g^\perp) = 0.
$$

Thus, integrating the equation satisfied by g_ϵ against $(v_g, v_\|)$, we deduce that the one satisfied by the current density $J_\epsilon = \int g_\epsilon v dv \left(= \begin{pmatrix} \int g_\epsilon v_\perp dv \\ \int g_\epsilon v_\| dv \end{pmatrix}\right)$ is the following:

$$
\begin{aligned}
&\partial_t J_\epsilon + \partial_{x_\|} \int v_\| \begin{pmatrix} v_g \\ v_\| \end{pmatrix} g_\epsilon dv + \nabla^\perp_{x_g}. \int E^1_{\epsilon,\perp}(t, x_g - v_g^\perp) \begin{pmatrix} v_g \\ v_\| \end{pmatrix} g_\epsilon dv\\
&= \int \begin{pmatrix} E^1_{\epsilon,\perp}(t, x_g - v_g^\perp) \\ 0 \end{pmatrix} g_\epsilon dv + \int \begin{pmatrix} 0 \\ E^1_{\epsilon,\|}(t, x_g - v_g^\perp) \end{pmatrix} g_\epsilon dv\\
&\qquad + \begin{pmatrix} 0 \\ E^2_\epsilon(t, x_{g,\|})\rho_\epsilon \end{pmatrix} + \frac{J_\epsilon^\perp}{\epsilon}.
\end{aligned}
\tag{7.11}
$$

We now assume that we deal with special monokinetic data of the form:

$$
g_\epsilon(t, x, v) = \rho_\epsilon(t, x)\delta_{v_\| = v_{\|,\epsilon}(t,x)}\delta_{v_g = 0}. \tag{7.12}
$$

This assumption is nothing but the classical "cold plasma" approximation together with the assumption that the transverse particle velocities are isotropically distributed (which is physically relevant, see [27]) : in other words, the average motion of particles in the perpendicular plane is only due to the advection by the electric drift $E^\perp$.

For the sake of readability, we denote by now $\nabla_{x_g} = \nabla_\perp$ and $\nabla_{x_\parallel} = \nabla_\parallel$. Note in particular that with these monokinetic data, we have in particular $J_\epsilon^\perp = 0$. Then we get formally the hydrodynamic model:

$$\begin{cases} \partial_t \rho_\epsilon + \nabla_\perp.(E_\epsilon^\perp \rho_\epsilon) + \partial_\parallel(v_{\parallel,\epsilon}\rho_\epsilon) = 0 \\ \partial_t(\rho_\epsilon v_{\parallel,\epsilon}) + \nabla_\perp.(E_\epsilon^\perp \rho_\epsilon v_{\parallel,\epsilon}) + \partial_\parallel(\rho_\epsilon v_{\parallel,\epsilon}^2) = -\epsilon\partial_\parallel \phi_\epsilon(t,x)\rho_\epsilon - \partial_\parallel V_\epsilon(t,x_\parallel)\rho_\epsilon \\ E_\epsilon^\perp = -\nabla^\perp \phi_\epsilon \\ -\epsilon^2 \partial_\parallel^2 \phi_\epsilon - \Delta_\perp \phi_\epsilon = \rho_\epsilon - \int \rho_\epsilon dx_\perp \\ -\epsilon \partial_\parallel^2 V_\epsilon = \int \rho_\epsilon dx_\perp - 1 \end{cases} \tag{7.13}$$

One can use the first equation to simplify the second one (the systems are equivalent provided that we work with regular solutions and that $\rho_\epsilon > 0$):

$$\begin{cases} \partial_t \rho_\epsilon + \nabla_\perp.(E_\epsilon^\perp \rho_\epsilon) + \partial_\parallel(v_{\parallel,\epsilon}\rho_\epsilon) = 0 \\ \partial_t v_{\parallel,\epsilon} + \nabla_\perp.(E_\epsilon^\perp v_{\parallel,\epsilon}) + v_{\parallel,\epsilon}\partial_\parallel(v_{\parallel,\epsilon}) = -\epsilon\partial_\parallel \phi_\epsilon(t,x) - \partial_\parallel V_\epsilon(t,x_\parallel) \\ E_\epsilon^\perp = -\nabla^\perp \phi_\epsilon \\ -\epsilon^2 \partial_\parallel^2 \phi_\epsilon - \Delta_\perp \phi_\epsilon = \rho_\epsilon - \int \rho_\epsilon dx_\perp \\ -\epsilon \partial_\parallel^2 V_\epsilon = \int \rho_\epsilon dx_\perp - 1. \end{cases} \tag{7.14}$$

Remarks 7.1. 1. *Notice here that we do not deal with the usual charge density and current density, since these ones are taken within the gyro-coordinates.*

2. *We mention that we could have considered the more general case:*

$$g_\epsilon(t,x,v) = \int_M \rho_\epsilon^\Theta(t,x)\delta_{v_\parallel = v_{\parallel,\epsilon}^\Theta(t,x)}\nu(d\Theta)\delta_{v_g=0} \tag{7.15}$$

where (M,Θ,ν) is a probability space which allows to model more realistic plasmas than "cold plasmas" and covers many interesting physical data, like multi-sheet electrons or water-bags data (we refer for instance to [2] and references therein). We will not do so for the sake of readability but we could deal with it with exactly the same analytic framework: the analogues of Theorems 2.1 and 2.2 identically hold. We get in the end the system:

$$\begin{cases} \partial_t \rho^\Theta + \nabla_\perp.(E^\perp \rho^\Theta) + \partial_\parallel(v_\parallel^\Theta \rho^\Theta) = 0 \\ \partial_t v_\parallel^\Theta + \nabla_\perp.(E^\perp v_\parallel^\Theta) + v_\parallel^\Theta \partial_\parallel(v_\parallel^\Theta) = -\partial_\parallel p(t,x_\parallel) \\ E^\perp = \nabla^\perp \Delta_\perp^{-1}\left(\int \rho^\Theta d\nu - \int \rho^\Theta dx_\perp d\nu\right) \\ \int \rho^\Theta(t,x) dx_\perp d\nu = 1. \end{cases} \tag{7.16}$$

As before, the equations are coupled through $x_\perp$ and here also through the new parameter Θ.

3. *Actually, the choice:*

$$g_\epsilon(t,x,v) = \rho_\epsilon(t,x)\delta_{v=v_\epsilon(t,x)} \tag{7.17}$$

leads to an ill-posed system. Indeed, we have to solve in this case equations of the form $v_\epsilon^\perp = v_{\epsilon,\perp}(t,x-v_\epsilon^\perp)$ where $v_{\epsilon,\perp}$ is the unknown. We can not say if this relation is invertible, even locally.

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References

[1] A.A. Arsenev. Existence in the large of a weak solution of Vlasov's system of equations. *Z. Vychisl. Mat. Mat. Fiz*, 15:136–147, 1975.

[2] N. Besse, F. Berthelin, Y. Brenier, and P. Bertrand. The multi-water-bag equations for collisionless kinetic modeling. *Kinet. Relat. Models*, 2(1):39–80, 2009.

[3] M. Bostan. The Vlasov-Poisson system with strong external magnetic field. Finite Larmor radius regime. *Asymptot. Anal.*, 61(2):91–123, 2009.

[4] Y. Brenier. Some conservation laws given by kinetic models. *Journées EDP*, pages 1–13, 1995.

[5] Y. Brenier. A homogenized Model for Vortex Sheets. *Arch. for Rational Mech. and Anal.*, 138:319–353, 1997.

[6] Y. Brenier. Convergence of the Vlasov-Poisson system to the incompressible Euler equations. *Comm. Partial Differential Equations*, 25:737–754, 2000.

[7] R. Caflisch. A simplified version of the abstract Cauchy-Kowalewski theorem with weak singularities. *Bulletin (New Series) of the American Mathematical Society*, 23(2):495–500, 1990.

[8] S. Cordier, E. Grenier, and Y. Guo. Two-stream instabilities in plasmas. *Methods Appl. Anal.*, 7(2):391–405, 2000.

[9] E. Frénod and A. Mouton. Two-dimensional Finite Larmor Radius approximation in canonical gyrokinetic coordinates. *Journal of Pure and Applied Mathematics: Advances and Applications*, 4(2):135–166, 2010.

[10] E. Frénod and E. Sonnendrücker. The Finite Larmor Radius Approximation. *SIAM J. Math. Anal.*, 32(6):1227–1247, 2001.

[11] P. Ghendrih, M. Hauray, and A. Nouri. Derivation of a gyrokinetic model. Existence and uniqueness of specific stationary solutions. *Kinet. and Relat. Models*, 2(4):707–725, 2009.

[12] F. Golse and L. Saint-Raymond. The Vlasov-Poisson system with strong magnetic field. *J. Math. Pures. Appl.*, 78:791–817, 1999.

[13] V. Grandgirard et al. Global full-f gyrokinetic simulations of plasma turbulence. *Plasma Phys. Control. Fusion*, 49:173–182, 2007.

[14] E. Grenier. Oscillations in quasineutral plasmas. *Comm. Partial Differential Equations*, 21(3-4):363–394, 1996.

[15] E. Grenier. Limite quasineutre en dimension 1. In *Journées "Équations aux Dérivées Partielles" (Saint-Jean-de-Monts, 1999)*, pages Exp. No. II, 8. Univ. Nantes, Nantes, 1999.

[16] Y. Guo and W.A. Strauss. Nonlinear instability of double-humped equilibria. *Annales de l'I.H.P., section C*, 12(3):339–352, 1995.

[17] D. Han-Kwan. On the confinement of a tokamak plasma. *SIAM J. Math. Anal*, 42(6):2337–2367, 2010.

[18] D. Han-Kwan. The three-dimensional finite Larmor radius approximation. *Asymptot. Anal.*, 66(1):9–33, 2010.

[19] D. Han-Kwan. Contribution à l'étude mathématique des plasmas fortement magnétisés. *PhD Thesis*, 2011.

[20] M. Hauray and A. Nouri. Well-posedness of a diffusive gyro-kinetic model. *To appear in Ann. IHP (Analyse Non Linéaire)*, 2011.

[21] T. Kato. On classical solutions of the two-dimensional nonstationary Euler equation. *Arch. Rational Mech. Anal.*, 25:188–200, 1967.

[22] Z. Lin, S. Ethier, T. S. Hahm, and W. M. Tang. Size Scaling of Turbulent Transport in Magnetically Confined Plasmas. *Physical Review Letters*, 88(19):195004–1–195004–4, May 2002.

[23] Andrew J. Majda and Andrea L. Bertozzi. *Vorticity and incompressible flow*, volume 27 of *Cambridge Texts in Applied Mathematics*. Cambridge University Press, Cambridge, 2002.

[24] C. Mouhot and C. Villani. On Landau damping. *Acta Math.*, 207(1):29–201, 2011.

[25] O. Penrose. Electrostatic instability of a uniform non-Maxwellian plasma. *Phys. Fluids*, 3:258–265, 1960.

[26] J. Simon. Compact sets in $L^p(0, T; B)$. *Ann. Mat. Pura. Appl.*, 146:65–96, 1987.

[27] P.L. Sulem. Introduction to the guiding center theory. *Topics in kinetic theory, Fields Inst. Commun., Amer. Math. Soc.*, 46:109–149, 2005.

[28] J. Wesson. Tokamaks. *Clarendon Press-Oxford*, 2004.

[29] V. I. Yudovič. Non-stationary flows of an ideal incompressible fluid. *Ž. Vyčisl. Mat. i Mat. Fiz.*, 3:1032–1066, 1963.